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Does star norm capture ℓ_1 norm?[1]

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Abstract

This paper pertains to the results obtained in [2] where the authors present a new norm, which they call the "$*$-norm", and conclude that minimizing the star norm results in good peak to peak disturbance rejection. The problem they treat reduces to solving a set of parameter dependent LMIs for standard synthesis problems in control and estimation. Unfortunately, we show using examples that minimizing the star norm does not necessarily imply good peak to peak disturbance rejection.
Keywords: *ℓ_1 norm, $*$ norm, extreme cases*

1. Introduction

The basic motivation of this work is to comment on the results and claims made by authors in [2]. We establish the following

1. The star norm is not a good approximation to ℓ_1 in that both the ratio between these two entities and the difference can be arbitrarily high.
2. The optimal star norm filter can amplify peak to peak disturbance rejection arbitrarily in comparison to its ℓ_1 counterpart.

For the sake of completion we briefly describe the results obtained in [2].

1.1. Analysis

The $*$-norm of the strictly proper, stable system

$$P = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad (1)$$

is given by

Definition 1 *Suppose that A in Equation(1) is stable, then the $*$ norm is defined as*

$$\|P\|_* = \min_{\alpha} \max_{x'Q_\alpha^{-1}x \leq 1} \|Cx\|$$

where Q is any symmetric positive definite matrix such that for some real number $\alpha > 0$, we have

$$AQ_\alpha + Q_\alpha A' + \alpha Q_\alpha + \frac{1}{\alpha}BB' \leq 0$$

and $\alpha \in (0, k)$ where $k = -2\max(real(\rho(A)))$, $\|.\|$ denotes the ℓ_2 norm.

1.2. Filtering

$$P = \begin{bmatrix} A & B \\ C_1 & 0 \\ C_2 & D \end{bmatrix}$$

Further let $BD' = 0$ and $DD' = I$ then the following theorem holds.

Theorem 1 *Suppose that (A, B) is controllable. Then the following statements are equivalent.*

1. *There exists a strictly proper, finite-dimensional, LTI filter F that renders $\|T_{zw}\|_* < \gamma$.*
2. *There exists a scalar $\alpha > 0$ and a symmetric matrix $Y_\alpha > 0$ such that Y_α is the stabilizing solution to the ARE*

$$AY_\alpha + Y_\alpha A' - \alpha Y_\alpha C_2' C_2 Y_\alpha + \alpha Y_\alpha + \frac{1}{\alpha}BB' = 0$$

and $\|C_1 Y_\alpha C_1'\|_2 < \gamma^2$.

2. Worst case $\frac{\|.\|_*}{\|.\|_{\ell_1}}$ ratio

Here, we show that the ratio between $*$ norm and the ℓ_1 norm can be infinite.

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Theorem 2 *Let*

$$\mathcal{P} = \{P \mid P \textit{ is strictly proper, stable}\}$$

then for any $M \in I\!R$ *there exists a* $P \in \mathcal{P}$ *such that*

$$\frac{\|P\|_*}{\|P\|_{\ell_1}} \geq M$$

We prove this by defining a set of plants parameterized by λ. Consider

$$\mathcal{P}_1 = \{P_\lambda | P_\lambda = \frac{1}{\sqrt{\lambda}} \frac{s - \lambda}{(s+\lambda)(s+\frac{1}{\lambda})}, \lambda \in \mathbb{R}_+\}$$

The result then follows by direct substitution. The result is quite obvious if one plots the corresponding reachable sets as shown in the figure. As the plot shows the reachable set estimated by the $*$ norm is exact along the major axis but is a poor estimate along the minor axis. The reason can be attributed to fitting an ellipsoid to the actual reachable set. Infact along the minor axis the estimate goes to a non-zero constant. Thus in order to get a large ratio all we need to do is to allign the output vector roughly along the minor axis while taking care that we do not lose observability while doing so.

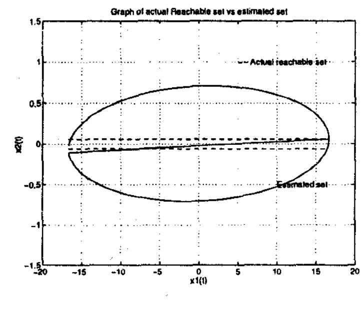
3. Filtering problem

Having done the above analysis we can intuitively see that control/estimation problems will suffer the same problems. To define precisely in what sense problems arise we follow on the lines of [1] and make the following definition.

Definition 2 *Consider a set of real rational SISO plants* $\mathcal{H}$, *and let* $\mathcal{L}(P, K)$ *denote the LFT of* $P, K \in \mathcal{H}$. *Let* $K_*(P)$ *denote the optimal star norm filter, and* $K_1(P)$ *the corresponding optimal* ℓ_1 *filter for a plant* $P \in \mathcal{H}$. *Then* $*$ *norm is said to be extreme compared to* ℓ_1 *in the* ℓ_1 *sense if*

$$\sup_P \frac{\|\mathcal{L}(P, K_*(P))\|_{\ell_1}}{\|\mathcal{L}(P, K_1(P))\|_{\ell_1}} = \infty$$

The reason one would expect $*$ norm to be extreme should be obvious. Consider for example a filtering problem where our objective is to minimize the star norm of the transfer function between exogenous inputs to the error in the hope that it will minimize the ℓ_1 norm. The transfer function for



example could be like the one considered in the previous theorem. Thus while the transfer function may have excellent peak to peak disturbance rejection properties, it may have a large star norm. This would mean the optimal star norm filter will differ markedly from that of an optimal ℓ_1 filter (which in the above case may have gain equal to zero). This will then result in a very ℓ_1 norm. The following theorem illustrates this particular case.

Theorem 3 *The* $*$ *norm is extreme compared to* ℓ_1 *in the* ℓ_1 *sense.*

Again we consider the following parameterized family of plants and deduce the result.

$$\mathcal{P}_2 = \{P_\lambda \mid P_\lambda \equiv \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\frac{1}{\lambda} & 1 & 0 \\ \frac{-2\lambda}{1-\lambda^2} & \frac{1}{\lambda}\frac{1+\lambda^2}{1-\lambda^2} & 0 & 0 \\ \frac{-2\lambda^{3/2}}{1-\lambda^2} & \frac{1}{\lambda^{1/2}}\frac{1+\lambda^2}{1-\lambda^2} & 0 & 1 \end{bmatrix}, \lambda \in \mathbb{R}_+\}$$

References

[1] S. R. Venkatesh, M.A. Dahleh *" Extreme Cases in ℓ_1 and $\mathcal{H}_\infty$ minimization," System and Control letters, 1994.*

[2] K Nagpal, J. Abedor, K. Poola, *An LMI approach to peak to peak gain minimization: Filtering and control, ACC 1994*