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Robust Performance for Fixed Inputs[1]

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Abstract

We address the problem of robust performance analysis when the exogenous input is assumed to be fixed and known . This differs from standard approaches in the literature which assume that the exogenous input is an unknown element in a class of norm bounded signals. When the performance is measured by the ℓ_∞ norm, and the nominal plant is perturbed by LTV perturbations of bounded ℓ_∞ induced norm, we propose upper and lower bounds for the measure of robust performance. Two upper bounds are derived. The first one can have direct application in robust performance synthesis problems. The second one provides a tighter bound. Both conditions are (usually) much less conservative than the condition resulting from assuming a worst case exogenous input. The necessary condition follows from the result of Khammash for robust steady state performance.

For certain classes of input signals, these upper and the lower bounds coincide, providing a necessary and sufficient condition.

1. Introduction

In most control problems the closed loop system has to be designed to achieve several conflicting objectives. It has to remain stable in the presence of unmodeled dynamics, and simultaneously reject unknown disturbances and track specific (given) commands.

So far, researchers efforts have concentrated mainly on the problem of robust stability/performance for disturbance rejection problems in which the disturbances are assumed to be unknown signals of bounded norm (energy or amplitude), [2 – 8]. In such a problem setup the robust performance problem turns out to be equivalent to an augmented robust stability problem. This equivalence is achieved by adding an extra uncertainty block between the performance output and the disturbance input. This so called "performance block" captures the induced norm performance specification.

In many cases, as in tracking problems, some of the exogenous inputs are known fixed signals and the objective of the closed loop system is to guarantee robust tracking or rejection of these fixed inputs in the presence of model uncertainty. In particular we consider the maximum peak amplitude as a performance measure.

Of course one can neglect this information about the input, assume that the input is just bounded but otherwise unknown and use the standard robust stability results. Clearly, this approach to guarantee robust performance in this case may be very conservative. In a broader sense, it is unclear how general is the equivalence between robust performance and robust stability for different performance specifications and/or input characteristics of practical interest. This motivate our approach that considers as separate issues the robust stability and the performance robustness. We will use this approach to develop (potentially) much less conservative results for robust performance with respect to fixed inputs. For the sake of brevity proofs have been omitted but will appear elsewhere.

2. Problem Statement

In this paper we consider discrete-time systems. The normed vector space denoted by ℓ_∞^n is the set of all vector valued sequences $\{x(k)\}$ with $x(k) \in \mathbb{R}^n$, such that

$$\|x\|_\infty \triangleq \sup_{k\geq 0} \max_{1\leq i\leq n} |x_i(k)|.$$

The ℓ_∞-induced norm, denoted by $\ell_\infty - ind$, of a linear dynamic system H is defined as the maximum peak-to-peak gain of the system:

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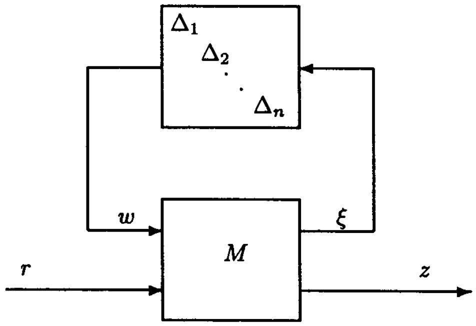

Figure 1: Standard Framework.

$$\|H\|_{\ell_\infty - ind} \triangleq \sup_{\|x\|_\infty \leq 1} \|Hx\|_\infty$$

The notation Hx here is intended as: H operates on a sequence x by convolution. If H is a linear time-invariant causal system then $\|H\|_{\ell_\infty - ind} = \|H\|_1$. The quantity $\|H\|_1$ is called the ℓ_1 norm of the system and for a $SISO$ system with impulse response h, it is defined as the absolute sum of the impulse response of H:

$$\|H\|_1 = \sum_{k=0}^{\infty} |h(k)|$$

For a system with p inputs and q outputs

$$\|H\|_1 = \max_{1 \leq i \leq q} \sum_{j=1}^{p} \|H_{ij}\|_1$$

We consider the problem being posed in the standard framework described in Figure 1. M is the generalized nominal system, assumed LTI, causal and internally stable. Δ is a diagonal LTV causal perturbation (Δ_i SISO system) with $\|\Delta\|_{\ell_\infty - ind} \leq 1$. We denote by $\underline{\Delta}$ the set of all such perturbations:

$$\begin{aligned} \underline{\Delta} = \ & \{\Delta \in LTV \text{ causal }, \mid \|\Delta\| \leq 1, \\ & \Delta = diag\{\Delta_1, \ldots \Delta_n\},\ \Delta_i \text{ SISO }\} \end{aligned}$$

where $\|\Delta\|$ stands for $\|\Delta\|_{\ell_\infty - ind}$.

Also denote by $\underline{\Delta}_{NC}$ the set of all diagonal LTV non-causal perturbations with $\|\Delta\| < 1$.

In Figure 1,r is the vector of p inputs that are fixed and known. For the sake of simplicity, z is assumed to be a single output on which we want to achieve a desired level of performance. The presence of the uncertainty in the nominal model is described by $\Delta \in \underline{\Delta}$.

We consider as a performance measure on z its maximum peak amplitude, $\|z\|_\infty$, and normalize the desired performance level to $\|z\|_\infty < 1$ by absorbing all weights into M. We assume that $r \in \ell^p_{\infty,e}$ i,e, for any finite interval $[0, T]$

$$\max_{1 \leq i \leq p} \sup_t |r_i(t)| < \infty.$$

This allows, among others, input signals with polynomial growth such as ramps or parabolas.

2.1. Problem Statement

The robust performance problem can be stated as follows:

Definition 2.1 *Given the inputs r we say the the system M has robust performance against $\underline{\Delta}$, or simply has robust performance, if the system is internally stable and $\|z\|_\infty < 1$ for all perturbations $\Delta \in \underline{\Delta}$.*

In order to address the problem above we introduce the following notation:

The system M is partitioned in the obvious way as:

$$M = \begin{bmatrix} M^s & M^{sp} \\ M^{ps} & M^p \end{bmatrix}.$$

Given a transfer function matrix G, $\hat{G}$ denotes the matrix of ℓ_1 norms of the SISO elements of G:

$$\hat{G} = \begin{bmatrix} \|G_{1,1}\|_1 & \cdots & \|G_{1,N}\|_1 \\ \vdots & & \vdots \\ \|G_{M,1}\|_1 & \cdots & \|G_{M,N}\|_1 \end{bmatrix}$$

3. A First Result

Define the Truncation operator of order N, acting on the space of all sequences as

$$\Pi_N(x) = \begin{cases} x(k) & for\ 0 \leq k \leq N \\ 0 & k > N \end{cases}$$

Define the Tail operator of order N as

$$T_N(x) \triangleq (I - \Pi_N)(x).$$

The necessary condition we are going to present is based on the condition for robust steady state performance presented in [1]. The steady state value of a vector valued sequence z is defined as

$$\|z\|_{ss} \triangleq \limsup_{k \to \infty} \|T_k(z)\|_\infty$$

A necessary condition for robust performance is given by the next theorem.

Theorem 3.1 *The system has robust performance only if*

$$\rho\left(\left[\begin{array}{cc} \hat{M}^s & \|M^{sp}r\|_{ss} \\ \hat{M}^{ps} & \|M^p r\|_{ss} \end{array}\right]\right) < 1 \qquad (1)$$

We recall the following property of nonnegative matrices:

Fact 3.1 *[9] For a nonnegative matrix A, (i.e., $a_{ij} \geq 0$), $\rho(A) \geq 1$ if and only if there exists a non trivial solution $x \geq 0$ to the system of linear inequalities $Ax \geq x$*

A sufficient condition for robust performance is given by the following theorem.

Theorem 3.2 *The system M has robust performance if*

$$\rho\left(\left[\begin{array}{cc} \hat{M}^s & \|M^{sp}r\|_\infty \\ \hat{M}^{ps} & \|M^p r\|_\infty \end{array}\right]\right) < 1 \qquad (2)$$

Note that in general, if $\|r\|_\infty \leq 1$

$$\rho\left(\begin{bmatrix} \hat{M}^s & \|M^{sp}r\|_\infty \\ \hat{M}^{ps} & \|M^p r\|_\infty \end{bmatrix}\right) \leq \rho\left(\begin{bmatrix} \hat{M}^s & \hat{M}^{sp} \\ \hat{M}^{ps} & \hat{M}^p \end{bmatrix}\right). \quad (3)$$

The RHS of (3) is the necessary and sufficient condition for robust performance against worst case bounded inputs.Therefore the new condition, although only sufficient for robust performance with the fixed input r, is potentially much less conservative than the condition available so far which guarantees robust performance for the *worst case* input.

4. Main Result

In this section we derive a less conservative (though more computationally expensive) condition to ensure robust performance than the one derived in Theorem 3.2. To simplify the treatment we consider the system M being 2×2, i.e., there is only one SISO Δ and we look for robust performance of the output z to the fixed input r. Generalization to problems of higher dimension will appear elsewhere. We assume zero initial conditions.

Before we present the new condition we need some preliminary results.

Definition 4.1 *Consider a linear time-invariant SISO system H with impulse response h, a fixed signal $\eta \in \ell_{\infty,e}$, and an unknown but bounded amplitude input signal w with $\|w\|_\infty \leq \gamma$. Define the following function of H, η and γ as*

$$f_\gamma(H,\eta) \triangleq \sup_{\|w\|_\infty \leq \gamma} \|Hw + \eta\|_\infty.$$

$f_\gamma(H,\eta)$ will be referred to as the worst-case performance for a level γ or simply as worst-case performance.

This measure is also important in the solution of the mixed ℓ_1/ℓ_∞ problem that will be presented elsewhere. The next theorem describes how the worst-case performance can be computed.

Theorem 4.1 *Under the setup of Definition 4.1,*

$$f_\gamma(H,\eta) = \sup_{N\geq 0}\left\{\gamma\sum_{k=0}^{N}|h(k)| \quad + |\eta(N)|\right\}. \qquad (4)$$

Note that for $\eta = 0$ the worst-case performance coincides with $\gamma\|H\|_1$.
Moreover the worst-case performance is bounded for any level $\gamma < \infty$, if and only if H is BIBO stable and $\eta \in \ell_\infty$.

Let $\eta = Gr$, r being the fixed input $r \in \ell_{\infty,e}$ and G a causal LTI system. From now on we are going to assume that: G is asymptotically stable and maps r into $\eta \in \ell_\infty$. We assume also that H is asymptotically stable. These assumptions imply that for any finite level γ , $f_\gamma(H,Gr) < \infty$.

In what follows we presents some properties that characterize the worst-case performance as a function of the level γ. Some of them are immediate consequences of the definition.

Let $z = Hw + Gr$. For any finite integer $L \geq 0$, and any finite level γ consider

$$\sup_{\|w\|_\infty \leq \gamma} \|\Pi_L(z)\|_\infty$$

Clearly

$$\sup_{\|w\|_\infty \leq \gamma} \|\Pi_L(z)\|_\infty = f_\gamma(\Pi_L(H), \Pi_L(Gr))$$

It is easy to see that

$$\lim_{L\to\infty} \sup_{\|w\|_\infty \leq 1} \|\Pi_L(z)\|_\infty = f_\gamma(H, Gr)$$

The equation above can be used as definition of the worst-case performance. This definition is often preferred since it allows us to define some important quantities.

Note that for L finite, the "sup" is actually a "max" in Equation (4). This implies that for some $N \leq L$

$$\max_{\|w\|_\infty \leq \gamma} \|\Pi_L(z)\|_\infty = \gamma\sum_{k=0}^{N}|h(k)| \quad + |(Gr)(N)|$$

For such N, (which is function of γ) let $\alpha_{L,\gamma}(H) = \sum_{k=0}^{N}|h(k)|$ and $\beta_{L,\gamma}(Gr) = |(Gr)(N)|$.

Define

$$\overline{H}_\gamma \triangleq \lim_{L\to\infty} \alpha_{L,\gamma}(H)$$

$$\overline{Gr}_\gamma \triangleq \lim_{L\to\infty} \beta_{L,\gamma}(Gr).$$

Note that $\{\max_{\|w\|\leq\gamma} \|\Pi_L(z)\|\}$ converges to $f_\gamma(H, Gr)$ as L goes to ∞, and $\{\alpha_{L,\gamma}(H)\}$ also converges as L goes to ∞, since $\alpha_{L,\gamma}(H) \leq \|H\|_1$ and is monotonically non-decreasing for all $L \geq 0$. Therefore $\{\beta_{L,\gamma}(Gr)\}$ is also a convergent sequence, and so the above limits are well defined.

From the above definition it follows that for any $\gamma \geq 0$,

$$f_\gamma(H, Gr) = \overline{H}_\gamma\gamma + \overline{Gr}_\gamma$$

In particular , denote $\overline{H}_\infty$ and $\overline{Gr}_\infty$ by

$$\begin{aligned} \overline{H}_\infty &= \lim_{L\to\infty}\lim_{\gamma\to\infty} \alpha_{L,\gamma}(H) \\ \overline{Gr}_\infty &= \limsup_{L\to\infty}\lim_{\gamma\to\infty} \beta_{L,\gamma}(Gr) \end{aligned} \tag{5}$$

It can be shown these limits are well defined.
Then we have the following results.

Lemma 4.1 *$\overline{H}_\infty = \|H\|_1$, and $\overline{Gr}_\infty \leq \|Gr\|_\infty$. If H has an Infinite Impulse Response then $\overline{Gr}_\infty = \|Gr\|_{ss}$.*

Lemma 4.2 *For fixed H, G and r, consider the worst-case performance as function of γ. Let $f_{\gamma_0}(H, Gr) = \overline{H}_{\gamma_0}\gamma_0 + \overline{Gr}_{\gamma_0}$ for some $\gamma_0 \geq 0$. Then*

$$f_\gamma(H, Gr) \geq \overline{H}_{\gamma_0}\gamma + \overline{Gr}_{\gamma_0}$$

for any $\gamma \geq 0$.

An implicit consequence of the above Lemma is that the worst-case performance is a convex function for $\gamma \geq 0$, as stated in the next Lemma.

Lemma 4.3 *For fixed H, G and r, consider the worst-case performance as function of γ. $f_\gamma(H, Gr)$ is monotonically nondecreasing and convex for $\gamma \geq 0$.*

Remark 4.1 *In particular it is true that $f_\gamma(H, Gr) \geq \overline{H}_\infty\gamma + \overline{Gr}_\infty$.*

An important result is given by next theorem:

Theorem 4.2 *If $\overline{H}_\infty < 1$ then $f_\gamma(H, Gr)$ is a contraction in γ for $\gamma \geq 0$, i.e., there exists $\delta < 1$ such that*

$$|f_{\gamma_1}(H, Gr) - f_{\gamma_2}(H, Gr)| \leq \delta|\gamma_1 - \gamma_2| \tag{6}$$

for all $\gamma_1 \geq 0, \gamma_2 \geq 0$. Moreover there exists a unique $\gamma^* \geq 0$ such that $\gamma^* = f_{\gamma^*}(H, Gr)$.

We are now ready to apply the worst-case performance measure to derive sufficient conditions for robust performance. As already mentioned we consider a two-inputs two-outputs system M.

$$\begin{bmatrix} \xi \\ z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w \\ r \end{bmatrix}$$

Theorem 4.3 *Define the set $\mathcal{C}$ as*

$$\mathcal{C} = \{w \mid \|w\|_\infty \leq \|\xi\|_\infty\}$$

If M is robustly stable and $\|z\|_\infty < 1$ for all $w \in \mathcal{C}$ then M has robust performance to the fixed input r.

Note that the above condition is necessary and sufficient for robust performance if $\Delta \in \underline{\Delta}_{NC}$ is allowed to be non-causal.

This is because the set $\mathcal{C}$ is exactly the one generated by such perturbations. It is also worthwhile recalling that the condition for robust stability of M is the same for both $\underline{\Delta}$ and $\underline{\Delta}_{NC}$.

We now present necessary and sufficient conditions for robust performance against the class of perturbations $\underline{\Delta}_{NC}$. The above theorem says that these conditions will be sufficient for robust performance against $\underline{\Delta}$.

Theorem 4.4 *Consider the following positive matrix obtained by computing (5) for each row of M.*

$$M_\infty = \begin{bmatrix} \overline{M_{11}}_\infty & \overline{M_{12}r}_\infty \\ \overline{M_{21}}_\infty & \overline{M_{22}r}_\infty \end{bmatrix}$$

Then the system M has robust performance against $\underline{\Delta}_{NC}$ if and only if

$$\begin{aligned} &1) \quad \rho(M_\infty) < 1 \\ &2) \quad f_{\gamma^*}(M_{21}, M_{22}r) < 1 \end{aligned} \tag{7}$$

where $\gamma^* \geq 0$ is the unique solution of the equation $\gamma = f_\gamma(M_{11}, M_{12}r)$.

Notice that condition (1) derives mostly from the robust stability specification, while condition (2) derives from the performance robustness specification.

4.1. Computation

From the property of contraction mappings we can derive a way to compute the condition for robust performance in (7). We use the following

Fact 4.1 *[10] Let X equipped with a norm $\|\cdot\|$ be a Banach space, and let $T : X \to X$ be a mapping for which there exists a fixed constant $\delta < 1$ such that*

$$\|T(x) - T(y)\| \leq \delta\|x - y\|, \qquad \forall\, x, y \in X$$

then for any $x \in X$, the sequence $\{x_n\}_1^\infty$ in X defined by

$$x_{i+1} = T(x_i); \qquad x_0 = x$$

converges to x^*. Moreover,

$$\|x^* - x_i\| \leq \frac{\delta^i}{1-\delta}\|T(x_0) - x_0\|$$

Clearly the above result applied to $f_\gamma(M_{11}, M_{12}r)$ allows us to compute γ^* to any degree of accuracy.

5. Examples

5.1. Disturbance rejection problem

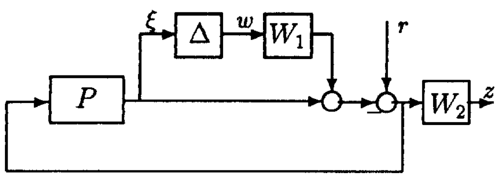

Figure 2: Disturbance Rejection Problem

We first start with a disturbance rejection problem, as shown in Figure 2. We want to check if the closed loop system rejects robustly a unit *step* input disturbance. The plant P has the following $z-transform$:

$$P(z) = \frac{z}{z^2 - 0.8z + .9}$$

The loop is closed by a unity feedback gain. The weight W_1 is used normalize the ℓ_∞*−induced* norm of the perturbation Δ. In this example $W_1 = .04$. $W2$ is used to normalize the output so that when $\|z\|_\infty < 1$ the performance is met. Typically W_2 should be a time-varying weight, although for the purposes of the example we have chosen the constant weight $W_2 = 0.8$. The matrix M is:

$$M = \begin{bmatrix} W_1 T & T \\ W_2 W_1 S & W_2 S \end{bmatrix}$$

where $S = (1+P)^{-1}$ is the sensitivity function, and $T = (1-S)$ is the complementary sensitivity function. First we check that necessary condition for robust performance from Theorem 3.1.

$$\rho\left(\begin{bmatrix} \|M_{11}\|_1 & \|M_{12}r\|_{ss} \\ \|M_{21}\|_1 & \|M_{22}r\|_{ss} \end{bmatrix}\right) = 0.9243 < 1. \qquad (8)$$

so that this condition is met. We now check the first sufficient condition presented in the paper in Theorem 3.2:

$$\rho\left(\begin{bmatrix} \|M_{11}\|_1 & \|M_{12}r\|_\infty \\ \|M_{21}\|_1 & \|M_{22}r\|_\infty \end{bmatrix}\right) = 1.3633 > 1$$

Unfortunately using this condition we are not able to conclude that M has robust performance. Now we wish to apply Theorem 4.4. Note that, since M_{11} and M_{21} are IIR systems, from Lemma 4.1 it follows that condition 1) of Theorem 4.4 is the same as has the necessary condition already verified in (8). Computing the rest of the condition in (7), we find that $\gamma^* = 1.1373$, and

$$f_{\gamma^*}(M_{21}, M_{22}r) = .9891 < 1.$$

Therefore M has robust performance against $\underline{\Delta}$. Note that the standard robust performance condition for the worst-case input in this case gives:

$$\rho\left(\begin{bmatrix} \|M_{11}\|_1 & \|M_{12}\|_1 \\ \|M_{21}\|_1 & \|M_{22}\|_1 \end{bmatrix}\right) = 10.74.$$

So that there is a very large difference between robust performance in the worst-case and robust performance for this fixed input.

5.2. Tracking problem

The scheme of the tracking problem we want to investigate is shown in Figure 3. In this case the plant is described as

$$P(z) = \frac{z}{z^2 - 0.8z + .99}$$

$W_1 = .001$, $W_2 = 6$ and T is the nominal complementary sensitivity.

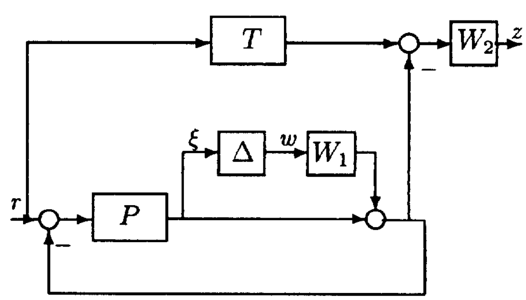

Figure 3: Tracking Problem

The problem is to check if, with the assumed level of perturbation, and a unit *step* input, the output of any possible perturbed closed loop system and the output of the nominal closed loop system differ in amplitude by less than $1/W_2$ for all times. By computing conditions (1), (2) and (7) we obtain respectively: 0.6618 , 0.9892 and 0.8548. In this case the robust performance test is already passed by using condition (2). This suggest that for the same family of perturbations better performance can be achieved ($W_2 > 6$). Again the standard robust stability test for the worst-case input is much more conservative, precisely 10.015 in this case.

6. Conclusions

We have defined a robust performance problem for fixed inputs, where the performance is measured by the maximum amplitude over time. The difficulty of the problem is due mainly to its time-varying nature. We have presented a necessary condition and two sufficient conditions for robust performance for fixed known inputs. Although some results have appeared for robust steady-state tracking of fixed inputs, the conditions presented here are, to our knowledge, the first results for robust tracking or robust disturbance rejection of fixed inputs for all times (i.e. including transient behavior). Both conditions are less conservative than using a standard worst case analysis. The second sufficient condition is less conservative than the first one we presented and in fact it is necessary if the perturbation is allowed to be non-causal. However, at the present time, the first condition seems to be more suitable for robust performance synthesis schemes, and this is a topic of current research.

7. Acknowledgments

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