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On robust control synthesis and analysis in a Hilbert space ☆

S.R. Venkatesh, A. Megretski, M.A. Dahleh *

Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Room 25-402, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

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Abstract

The motivation for this paper stems from the need to develop a uniform framework for addressing problems in identification and robust control. System identification for infinite-dimensional Hilbert spaces has been addressed earlier by the authors. System identification set in an Hilbert space results in uncertain models where the description of non-parametric error is typically a ball belonging to the Hilbert space. The scope of this paper is to complement these results — develop robust control-synthesis and analysis results — for some special, yet, important cases. In this paper we derive a convex parameterization of robustly stabilizing controllers for LTI discrete-time systems defined on Hilbert spaces. The perturbations are of rank-one type having both real-parametric and non-parametric components. The parameterization allows for imposing other constraints to obtain meaningful performance from the controller. Analysis tools are also developed for robust stability under SISO block-diagonally structured perturbations. The robustness analysis problem reduces to a finite-dimensional LMI verification which makes the procedure extremely efficient.

1. Notation

Let ℓ be the space of real-valued infinite sequences supported on Z^+. Let ℓ_1 and ℓ_2 denote the subspaces of ℓ of absolutely summable sequences and square summable sequences, respectively. The λ transform of any sequence $H = (h(k)) \in \ell_1$ is given by

$$H(\lambda) \doteq \sum_{k=0}^{\infty} h(k)\lambda^k, \quad \lambda \in \mathbb{C}.$$

The space ℓ_1 can also be identified with the disc algebra of bounded analytic functions on the closed unit disc. For the sake of notational simplicity we do not make any distinction between a sequence in ℓ_1 and its λ transform.

We will be concerned with subspaces of ℓ that have a Hilbert space structure. We denote such spaces as $\mathscr{H}(r)$ where $r = (r(k))$ is an element of the set, $\mathscr{S} \subset \ell$. The set, $\mathscr{S}$, is given as follows:

$$\mathscr{S} \doteq \{r = (r(k)) \in \ell\colon r(k) \geqslant k\log(k) + 1\}.$$

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* Corresponding author. Tel.: +1-617-253-3892.

E-mail address: dahleh@lids.mit.edu (M.A. Dahleh)

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The norm of any sequence, $H \in \mathscr{H}(r)$, is defined by

$$\|H\|^2_{\mathscr{H}(r)} \doteq \sum_{k=0}^{\infty} r(k)h^2(k).$$

The inner product between any two sequences, $H_1 = (h_1(k)) \in \mathscr{H}(r)$ and $H_2 = (h_2(k)) \in \mathscr{H}(r)$, is defined by

$$\langle H_1, H_2 \rangle_{\mathscr{H}(r)} \doteq \sum_{k=0}^{\infty} r(k)h_1(k)h_2(k), \quad (r(k)) \in \mathscr{S}.$$

The motivation for the class arises from the fact that $\mathscr{H}(r) \subset \ell_1$, i.e., if an infinite sequence $(h(k)) \in \mathscr{H}(r)$, then $(h(k)) \in \ell_1$. Thus, any bounded sequence $(h(k))$ defined by the $\mathscr{H}(r)$ norm is a kernel of some BIBO stable convolution operator. As a point of digression, when $r(k) = k^2 + 1$, the space is referred to as the Hardy–Sobolov space usually denoted by the symbol $\mathscr{H}^{2,1}$. The Hardy–Sobolov norm in the frequency domain is obtained by summing the energy of the spectrum with the energy of the derivative of the spectrum.

Finally, for any norm $|\cdot|$ on $\mathbb{R}^m$, we define the dual norm $|\cdot|^d$ by

$$|x|^d \doteq \max\{x^{\mathrm{T}} y\colon |y| \leqslant 1\}.$$

2. Introduction

It is widely perceived that system-identification and robust control present a fundamental dichotomy. Robust control provides a framework for addressing control design and analysis issues when faced with uncertainty in the plant dynamics. However, it is often the case that the description of uncertainty is not consistent with models obtained from system identification. This subject matter has received wide attention from as early as late 1980s culminating in devoting an entire issue of the IEEE Transactions on Automatic Control to this topic (see [3]).

With these factors serving as the backdrop a new formulation for the system identification problem was presented in [6–8]. Our formulation addressed problems arising out of identification of infinite-dimensional systems from finite-noisy data and we will give a brief account of it here for the sake of completion. The class of infinite-dimensional LTI systems ($\mathscr{T}$) cannot be uniformly approximated by a finite-dimensional space. Nevertheless, we represent our *prejudice* by selecting a finitely parameterized set of models ($\mathscr{G}$) from which an estimate of the original system will ultimately be drawn. The motivation is that it is possible to only estimte a finite number of parameters from noisy finite data and the objective should be to estimate that model from the model parameterization that minimizes the unmodeled dynamics. The outcome of the identification problem will result in estimates of the model, the parametric and non-parametric errors.

In general, solutions to such identification problems are computationally cumbersome. However, in a Hilbert space setting, it is possible to significantly reduce the computational complexity. By appealing to the duality theorem the infinite-dimensional system, $T \in \mathscr{T}$, can be uniquely decomposed into a model G belonging to the subspace $\mathscr{G}$, the unmodeled dynamics minimizer, and the residual, $T - G$, the space of systems orthogonal to $\mathscr{G}$. This helps in the construction of annihilating filters that when applied to the output virtually strip away the unmodeled error in the input–output equation. This forms the basis for the solution to the identification problem. The solution can be implemented recursively and the corresponding computational complexity is no more than that of RLS. Although, the solution to the identification problem is also possible in the ℓ_1 topology, the computational complexity is significantly higher.

These facts serve as a motivation to pursue robust control synthesis and analysis for Hilbert spaces and we identify some of the salient issues here. In short, the residual error dynamics, resulting from the identification procedure, is an arbitrary element belonging to the set of all norm (defined on the Hilbert space) bounded systems. Now robustness analysis generally examines the stability of a feedback interconnection of a system and an arbitrary perturbation belonging to a set of stable systems. The set of perturbations are required to be stable because the proof technique relies on a continuity of poles argument (see [11]). The space $\mathscr{H}^2$ of systems, with bounded impulse response energy admits unstable systems making it unsuitable for stability analysis using traditional robust control techniques. Therefore, in order to maintain consistency between robust control and identification, we have to restrict the class of systems to a space that has a Hilbert space structure and yet satisfies the requirements of robust control (i.e., where norm boundedness implies stability). In general, an appropriately weighted energy norm suffices. This is indeed the motivation for the space of functions denoted by the symbol, $\mathscr{H}(r)$, in Section 1.

We will now enumerate the organization of the paper. The robust control problem is discussed in Section 3. We restrict ourselves to perturbation that are of rank-one (perturbations that can either be written as multi-input–single-ouput or single-output–multi-input). Following on the lines of [5] a convex parameterization of all robustly stabilizing controllers is derived. The methodology adopted is general enough to extend it to mixed rank-one perturbations with real-parameter uncertainties. It turns out that the optimal robust controller even for the SISO problem is in general infinite dimensional. This is in direct contrast to both the $\mathscr{H}_\infty$ and ℓ_1 problems.

The robust analysis problem is discussed in Section 4. We develop tools for analysis for robust stability in the face of diagonally structured SISO block perturbations belonging to the Hilbert space. Sufficient conditions for robust stability is derived by employing the well-known $\mathscr{S}$ procedure (see [4]). The robust analysis problem reduces to verifying a finite-dimensional LMI thus making the problem computationally efficient. The solution to the problem leads us to developing new sufficient conditions for robust stability for structured real-parameter uncertainties.

3. Synthesis problem

In this section we will formulate the synthesis problem and give a convex parameterization of all robustly stabilizing controllers. The setup is as shown in Fig. 3. In this configuration G is a known LTI system representing, as usual, the nominal system and K represents the control system. As is now typical, we represent the perturbation Δ, which denotes the system uncertainty, in the form of a feedback loop as shown in Fig. 3. Specifically, w the input to Δ, is a scalar signal. The results in the paper also hold when the output, z, is a scalar and for the sake of brevity we assume that w is a scalar signal. It follows that the uncertainty Δ is spatially structured, belonging to the set $\boldsymbol{\Delta}$ given by

$$\boldsymbol{\Delta} = \{\Delta\colon \Delta = [\delta_1, \dots, \delta_L, \Delta_{L+1}, \dots, \Delta_{L+F}]\},\ |(\delta_1, \dots, \delta_L)| \leqslant 1;\ \|\Delta_i\|_{\mathscr{H}(r_i)} \leqslant 1, \quad \forall i, \tag{1}$$

where δ_i is a scalar real-parametric perturbation and Δ_k is a non-parametric SISO perturbation. A note of caution in our notation. We denote the impulse response of $\Delta \in \mathscr{H}(r)$ by $(\delta(k))$ and this should not be confused with the real-parameter uncertainties which are indexed by means of subscript. We point out that we allow different weight sequences, $r \in \mathscr{S}$, for each non-parametric perturbation amounting to a different norm for each output channel. This will be particularly useful in the next section when we consider mixed real-parametric and non-parametric perturbations. With these preliminaries we will now describe obtaining a convex parameterization of all controllers K that simultaneously stabilize the system for all $\Delta \in \boldsymbol{\Delta}$. The result is obtained by a combination of well-known ideas and we follow the development in [5] in deriving these results. First, we need a definition for stability.

Definition 1. A system $T : u \to y$ that takes inputs, $u \in \ell$, to outputs, $y \in \ell$, is said to be stable if it satisfies the conditions for BIBO stability (see [1]). For an LTI system T the conditions for BIBO stability are met if and only if $T \in \ell_1$. Equivalently, an LTI system T is stable if it belongs to the disc algebra of bounded analytic functions on the closed unit disc.

With this definition for stability as a backdrop, consider the setup of Fig. 1 and note in particular that w is a scalar. From Youla parameterization [10] of stabilizing controllers, we may assume that the admissible transfer functions, $\mathscr{T}_{zw}$, from w to z are of the form

$$\mathscr{T}_{zw} = T_1 + T_2 Q, \tag{2}$$

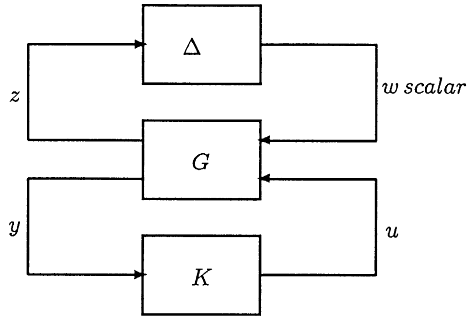

Fig. 1. Uncertain control system.

where $T_1, T_2 \in \ell_1^{m+r}$ are fixed and Q is any transfer function in ℓ_1. As the uncertainty loop $w = \Delta z$ is closed, the system is robustly stable if and only if

$$(1 - \Delta(T_1 + T_2 Q))^{-1} \in \ell_1, \quad \forall \Delta \in \boldsymbol{\Delta}, \tag{3}$$

where $\boldsymbol{\Delta}$ is as in Eq. (1). The problem is to find a convex parameterization of all $Q \in \ell_1$ such that Eq. (3) is satisfied.

3.1. SISO case

We first derive the solution for the simple case when there is only one block of perturbation, i.e., let

$$\boldsymbol{\Delta}_0 = \{\Delta \in \mathscr{H}(r): \|\Delta\|_{\mathscr{H}(r)} \leqslant 1, \, r \in \mathscr{S}\}.$$

We have the following theorem:

Theorem 1. *Suppose $T_1, T_2 \in \ell_1$. Then the following two conditions on the function $Q \in \ell_1$ are equivalent.*
1. *$Q \in \ell_1$ and*

$$[1 - \Delta(T_1 + T_2 Q)]^{-1} \in \ell_1, \quad \forall \Delta \in \boldsymbol{\Delta}_0. \tag{4}$$

2. *$Q = \beta/\alpha$ for some $\alpha, \beta \in \ell_1$ satisfying*

$$\left\| W' \Re \begin{pmatrix} T_1\alpha + T_2\beta \\ \mathrm{i}(T_1\alpha + T_2\beta) \end{pmatrix} (\exp(\mathrm{i}\omega)) \right\|_2 < \Re\,(\alpha(\exp(\mathrm{i}\omega))), \quad \omega \in [-\pi, \pi] \tag{5}$$

where $W(\cdot)$ is a continuous matrix function of $\omega \in [-\pi, \pi]$.

Proof. The proof follows in several steps. First, we determine the image $\Delta \in \mathscr{H}(r)$ on the boundary of the unit disc. The image turns out to be an ellipse at every point on the boundary of the unit disc as seen in Fig. 2. This will lead us to establish the equivalence between the two conditions.

Step 1 (*Image of $\boldsymbol{\Delta}_0$ in the frequency domain*): Let the impulse response of $\Delta \in \boldsymbol{\Delta}_0$ be given by $(\delta(k))$. At any point on the unit disc, say $\exp(\mathrm{i}\omega)$, the image is given by the following set in the complex plane:

$$\left\{ \sum_{k=0}^{\infty} \delta(k)(\cos(k\theta) + \mathrm{i}\sin(k\theta)) \in \mathbb{C}: \Delta \in \boldsymbol{\Delta}_0 \right\}.$$

Note that the set is bounded follows from the fact that $\sup_{\Delta \in \boldsymbol{\Delta}_0} \|\Delta\|_{\ell_1}$ is bounded. Equivalently, we may characterize the real and imaginary parts as follows:

$$\Phi(\omega) = \left\{ \begin{bmatrix} x_1(\omega) \\ x_2(\omega) \end{bmatrix} Kz \in \mathbb{R}^2: \|z\|_{\ell_2} \leqslant 1 \right\}, \tag{6}$$

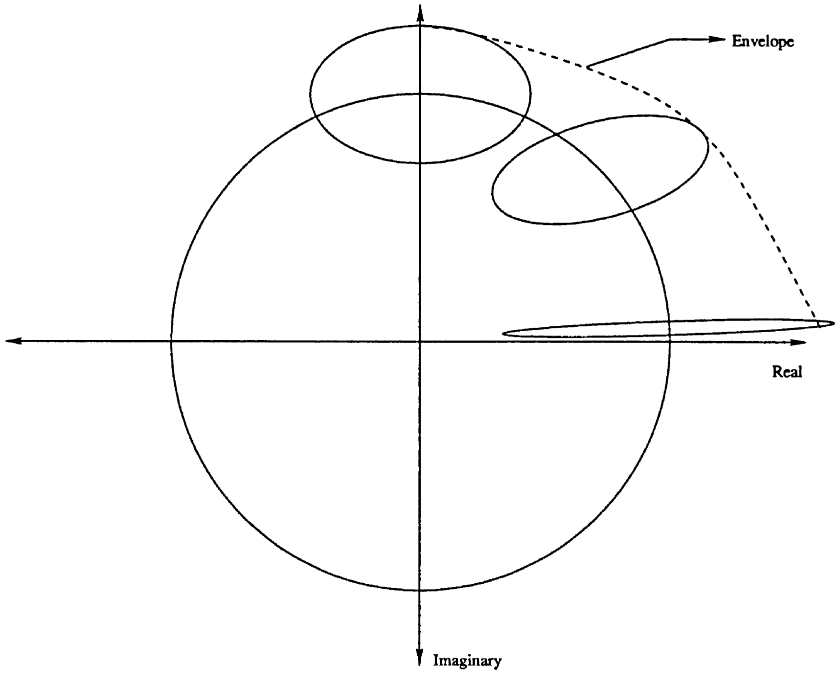

Fig. 2. Image of Δ is an ellipse at every point on the disc.

where

$$x_1(\omega) = (1, \cos(\omega), \cos(2\omega), \ldots),$$

$$x_2(\omega) = (0, \sin(\omega), \sin(2\omega), \ldots),$$

$$K = \mathrm{diag}[1/r(0), 1/r(1), 1/r(2), \ldots, 1/r(k), \ldots].$$

Next, we point out that:

$$\mathscr{L}(\omega) = \begin{bmatrix} x_1(\omega) \\ x_2(\omega) \end{bmatrix} K$$

is a finite-rank operator on ℓ_2 and we can therefore obtain a singular value decomposition of it:

$$\mathscr{L}(\omega) = (U\Sigma V')(\omega),$$

where $\Sigma = \mathrm{diag}[\sigma_1(\omega), \sigma_2(\omega)]$. We now let W be a weighting function over $\omega \in [-\pi, \pi]$ defined as

$$W(\omega) = (U\Sigma)(\omega). \tag{7}$$

It follows that the set $\Phi(\omega)$ can be equivalently characterized as:

$$\Phi(\omega) = \left\{ W(\omega) \begin{bmatrix} \eta \\ \xi \end{bmatrix} : \eta, \xi \in \mathbb{R}, \ \|[\eta, \xi]\|_2 \leqslant 1 \right\}. \tag{8}$$

In order to apply the results in [5] we will have to prove that the weighting function, W, is continuous as a function of frequency. This is not difficult upon observation of the fact that the singular values and vectors are amplitude and directions of the major and minor axes of the ellipsoid. These quantities can be computed analytically which we present here for the sake of completion. For the sake of notational simplicity, let

$$C = \sum_{k=0}^{\infty} \frac{\cos(2k\omega)}{2(r(k))}, \qquad S = \sum_{k=0}^{\infty} \frac{\sin(2k\omega)}{2(r(k))}.$$

Then, the amplitude of the major and minor axis as a function of frequency is given by

$$\Sigma^2 = \tfrac{1}{2}(\mathrm{diag}[1 + \sqrt{C^2(\omega) + S^2(\omega)}, 1 - \sqrt{C^2(\omega) + S^2(\omega)}])$$

and the singular vectors $U(\omega) = [u_1/\|u_1\|_2, u_2/\|u_2\|_2]$ as a function of frequency are given by

$$u_1 = \begin{bmatrix} \frac{\sqrt{C^2(\omega)+S^2(\omega)}+C(\omega)}{S(\omega)} & 1 \end{bmatrix}', \qquad u_2 = \begin{bmatrix} \frac{-\sqrt{C^2(\omega)+S^2(\omega)}+C(\omega)}{S(\omega)} & 1 \end{bmatrix}'.$$

This shows that the weight, W, is continuous with respect to frequency.

Step 2 (*Equivalence between the two conditions of the theorem*): The following statements are equivalent:

(A) $(1 - \Delta(T_1 + T_2Q))^{-1} \in \ell_1$ for all $\Delta \in \boldsymbol{\Delta}_0$.

(B) $(1 - \Delta(T_1 + T_2Q))(\exp(\mathrm{i}\omega)) \neq 0$ for $\omega \in [-\pi, \pi]$ and $\Delta \in \boldsymbol{\Delta}_0$.

(C)

$$\left(1 - [\eta, \xi]W' \begin{bmatrix} T_1 + T_2Q \\ \mathrm{i}(T_1 + T_2Q) \end{bmatrix} (\exp(\mathrm{i}\omega))\right) \neq 0 \tag{9}$$

for $\eta, \xi \in \mathbb{R}$ and $\|[\eta, \xi]\|_2 \leqslant 1$.

(D) There is a rational function $Q \in \ell_1$ and there exists a rational function $\alpha \in \ell_1$ that satisfies

$$\Re\left\{\left(1 - [\eta, \xi]W' \begin{bmatrix} T_1 + T_2Q \\ \mathrm{i}(T_1 + T_2Q) \end{bmatrix} (\exp(\mathrm{i}\omega))\right) \alpha(\exp(\mathrm{i}\omega))\right\} > 0 \tag{10}$$

for $\Delta \in \boldsymbol{\Delta}_0, \omega \in [-\pi, \pi]$ and $\|[\eta, \xi]\|_2 \leqslant 1$.

(E) There exist rational functions $\alpha, \beta \in \ell_1$ such that $Q = \beta/\alpha$ and

$$\Re\,(\alpha(\exp(\mathrm{i}\omega))) - [\eta, \xi]\Re\left(W' \begin{bmatrix} T_1\alpha + T_2\beta \\ \mathrm{i}(T_1\alpha + T_2\beta) \end{bmatrix} (\exp(\mathrm{i}\omega))\right) > 0 \tag{11}$$

for $\Delta \in \boldsymbol{\Delta}_0$, $\omega \in [-\pi, \pi]$ and $\|[\eta, \xi]\|_2 \leqslant 1$.

The fact that (E) is equivalent to the second statement of Theorem 1 and (A) implies (B) is straightforward. We need to prove that (B) implies (A) which we do so using the usual continuity of poles argument (see [11]). We present it here for the sake of completion. Consider the set of solutions to the following equation:

$$(1 - \varepsilon\Delta(T_1 + T_2Q))(\lambda) = 0, \quad \Delta \in \boldsymbol{\Delta}_0, \ \lambda \in \mathbb{C} \tag{12}$$

as a function of the parameter $\varepsilon \in [0, 1]$ for a fixed $\Delta \in \boldsymbol{\Delta}_0$. We know that for small enough ε the above equation is satisfied only for $|\lambda| > 1$, i.e. $(1 - \varepsilon\Delta(T_1 + T_2Q))^{-1}$ is analytic on the closed unit disc for small enough ε. For a fixed Δ, solutions to Eq. (12) are continuous with respect to ε. Now, if (A) does not hold for some $\Delta \in \boldsymbol{\Delta}_0$ but (B) holds, it follows that, Eq. (12) has a solution for some $|\lambda| < 1$ and $\Delta \in \boldsymbol{\Delta}_0$. But now by continuity there is an $\varepsilon \in [0, 1]$ such that for some $|\lambda| = 1$ Eq. (12) holds, contradicting (B). That (B) and (C) are equivalent follows from Eqs. (6) and (8). To prove that (C) and (D) are equivalent we note that the set

$$\left\{\eta, \xi \in \mathbb{R}\colon \left(1 - [\eta, \xi]W' \begin{bmatrix} T_1 + T_2Q \\ \mathrm{i}(T_1 + T_2Q) \end{bmatrix} (\exp(\mathrm{i}\omega))\right), \ \|[\eta, \xi]\|_2 \leqslant 1\right\} \tag{13}$$

is convex at every ω. Therefore, there is a complex valued function, $\alpha(\exp(\mathrm{i}\omega))$, satisfying Eq. (13). That such a function is continuous and can be approximated by a real rational function in ℓ_1 follows exactly as in [5]. The equivalence between (D) and (E) follows by change of variables ($Q = \beta/\alpha$). Note that β/α is stable follows from the fact that the choice $\eta = 0$, $\xi = 0$ implies that $\mathrm{Re}(\alpha(\exp(\mathrm{i}\omega))) > 0$. We can see this easily by making the usual conformal map, $z = C(s)$, from the unit-disc to the imaginary axis. Then $\alpha(C(s))$ is a strictly positive real transfer function on the imaginary axis. This implies that $\alpha(C(s))$ is stable minimum-phase transfer function.

3.2. Extensions

We now extend the results of Theorem 1 to the general rank-one perturbations belonging to the set, $\boldsymbol{\Delta}$, given in Eq. (1). The general condition for robust stability in this case is

$$(1-[\delta',\Delta_1,\dots,\Delta_m](T_1+T_2Q))^{-1}\in\ell_1,\quad \forall\Delta=[\delta',\Delta_1,\dots,\Delta_m]\in\boldsymbol{\Delta}. \tag{14}$$

Again T_1+T_2Q describes the nominal closed-loop transfer function from w to z and the decomposition:

$$[T_1 \quad T_2]=\begin{bmatrix} T_1^{\delta} & T_2^{\delta}\\ T_1^{\Delta_1} & T_2^{\Delta_1}\\ \vdots & \vdots\\ T_1^{\Delta_m} & T_2^{\Delta_m}\end{bmatrix}\in\ell_1^{m+r} \tag{15}$$

corresponds to the decomposition taken with respect to the perturbation, $\Delta=[\delta',\Delta_1,\dots,\Delta_m]$. We have the following corollary for the synthesis problem which follows in a straightforward fashion from Theorem 1.

Corollary 1. *The following two conditions on the rational function Q are equivalent*:

1. *There is a rational function $Q\in\ell_1$ and*
$$[1-\Delta(T_1+T_2Q)]^{-1}\in\ell_1,\quad \Delta\in\boldsymbol{\Delta}, \tag{16}$$
where $\boldsymbol{\Delta}$ is given by Eq. (1).
2. *$Q=\beta/\alpha$ for some rational functions $\alpha,\beta\in\ell_1$ satisfying*
$$|\Re\,(T_1^{\delta}\alpha+T_2^{\delta}\beta)(\exp(\mathrm{i}\omega))|^{\mathrm{d}}+\sum_{i=1}^{m}\left\|W'\Re\begin{pmatrix}T_1^{\Delta_i}\alpha+T_2^{\Delta_i}\beta\\ \mathrm{i}(T_1^{\Delta_i}\alpha+T_2^{\Delta_i}\beta)\end{pmatrix}(\exp(\mathrm{i}\omega))\right\|_2<\Re\,(\alpha(\exp(\mathrm{i}\omega))), \tag{17}$$
where $\omega\in[-\pi,\pi]$, $W(\cdot)$ is a continuous matrix function of $\omega\in[-\pi,\pi]$ and $|\cdot|^{\mathrm{d}}$ is the norm on the dual of the space of real-parameter uncertainties.

Again as in [5] we can also extend these results to the case of coprime-factor descriptions of uncertain systems for single-input–multi-output (SIMO) and multi-input–single-output (MISO) type systems. The results follow in a straightforward way as an extension to Corollary 1 and is omitted here.

The solution methodology presented here requires solving infinite-dimensional optimization problems. As in [5] we propose using the "Ritz" method consisting of solving the problem over larger and larger finite-dimensional spaces. Here, this means that we solve for feasible solutions to Eq. (17) over, α,β. Dual formulations for these class of problems is currently being pursued in order to give strict accuracy bounds which can be used as a stopping criterion.

4. Robustness analysis for structured uncertainty

In this section we will derive conditions for robust stability for diagonally structured SISO blocks of systems belonging to the Hilbert space. Consider the feedback configuration shown in Fig. 3. We assume that both G and Δ are LTI stable discrete-time systems. The system G is known and the perturbation Δ belongs to the following collection:

$$\boldsymbol{\Delta}=\{\Delta\colon\ \Delta=\mathrm{diag}[\Delta_1,\Delta_2,\dots,\Delta_n],\ \Delta_i\,\mathrm{SISO},\ \|\Delta\|_{\mathscr{H}(r_i)}\leqslant 1,\ r_i\in\mathscr{S},\ i=1,2,\dots,n\}. \tag{18}$$

We point out that the norm is different on each channel, characterized by different weighting sequences. The two issues of interest in an interconnection of two system is well-posedness and stability. Well-posedness in the linear setting implies that $(I-G\Delta)^{-1}$ is causally invertible on the space ℓ. In addition, the interconnection is stable as long as the inverse is also bounded, i.e.,

$$\sum_{t=0}^{T}\|v(t)\|_2^2+\|w(t)\|_2^2\leqslant\sum_{t=0}^{T}\|e(t)\|^2+\|f(t)\|_2^2,\quad \forall T,\ f(\cdot)\in\ell_{2e}. \tag{19}$$

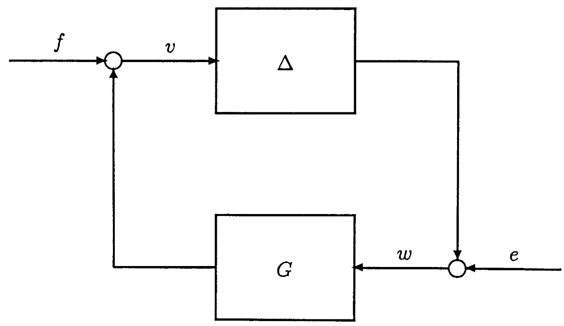

Fig. 3. Robust-stability for structured perturbations.

Alternatively, since G and Δ are linear the condition simplifies to

$$\det(I - G\Delta)(\exp(\mathrm{i}\omega)) \neq 0, \quad \forall\omega \in [-\pi, \pi], \ \Delta \in \mathbf{\Delta}. \tag{20}$$

In the $\mathscr{H}_\infty$ situation the structured block-diagonal perturbation (see [2]) is typically dealt in the following manner. One observes that mapping G to $D^{-1}GD$ with diagonal frequency-dependent scales D does not change the condition for stability because $\Delta \to D^{-1}\Delta D$ is a one-to-one mapping. This then forms the basis for several algorithms to check stability wherein a search for the minimum of the maximum singular value of $D^{-1}GD$ gives a good sufficient condition for stability. In our situation, as seen in the previous section the image of each block perturbation is an ellipse. It is difficult to find a transformation on G as in $\mathscr{H}_\infty$ situation that does not change the condition for stability. The principle problem is that, unlike $\mathscr{H}_\infty$ or ℓ_1, the Hilbert spaces under consideration are not Banach algebras and so even though $H_1, H_2 \in \mathscr{H}(r)$, we cannot guarantee that $H_1H_2 \in \mathscr{H}(r)$. However, it should be noted that this property alone does not suffice. Indeed in the case of ℓ_1 a diagonal frequency-dependent scaling is in general inadmissible because the mapping in general is not one to one (for MIMO unstructured blocks).

To exploit the structure of the perturbations we apply the well-known S-procedure technique (see [4] and references therein) for this problem. We state the salient features here for the sake of completion.

The application of this technique, for our situation, requires "modeling" the uncertain elements by means of quadratic constraints (QCs). Consider two signals, $v \in \ell_2^n$ and $w \in \ell_2^n$ and their corresponding fourier transforms $\hat{v}(\omega)$ and $\hat{w}(\omega)$ at some frequency $\omega \in [-\pi, \pi]$. Let

$$\hat{v}_r(\omega) = \Re(\hat{v}(\omega)), \quad \hat{v}_i(\omega) = \Im(\hat{v}(\omega)), \quad \hat{w}_r(\omega) = \Re(\hat{w}(\omega)), \quad \hat{w}_i(\omega) = \Im(\hat{w}(\omega)).$$

Two signals $u \in \ell_2^n$ and $w \in \ell_2^n$ are said to satisfy a QC defined by $\Pi(\omega)$ if

$$\begin{bmatrix} \hat{v}_r(\omega) \\ \hat{v}_i(\omega) \\ \hat{w}_r(\omega) \\ \hat{w}_i(\omega) \end{bmatrix}^{\mathrm{T}} \Pi(\omega) \begin{bmatrix} \hat{v}_r(\omega) \\ \hat{v}_i(\omega) \\ \hat{w}_r(\omega) \\ \hat{w}_i(\omega) \end{bmatrix} \geqslant 0, \tag{21}$$

where $\Pi(\omega)$ is any symmetric matrix valued function taking values in $\mathbb{R}^{4n\times 4n}$. These QCs are used to describe relations between signals of perturbation Δ at each frequency.

The collection of LTI stable sytems $\mathbf{\Delta}$ is said to satisfy the QC at frequency ω defined by $\Pi(\omega)$ if Eq. (21) holds for $w = \Delta v$. Once the uncertainty is modeled with QCs we can derive a robust stability result as follows which is based on $\mathscr{S}$ procedure.

Theorem 2. *Let G and Δ be LTI stable discrete-time systems, interconnected as in Fig.* 3, *with Δ having the structure defined by Eq.* (18). *Further, let the feedback interconnection be well-posed. Suppose, the quadratic constraint defined by $\Pi(\omega)$ at every frequency is satisfied by every $\Delta \in \mathbf{\Delta}$ and there is an $\varepsilon > 0$ such that*

$$\begin{bmatrix} L_G \\ I \end{bmatrix}^{\mathrm{T}} \Pi(\omega) \begin{bmatrix} L_G \\ I \end{bmatrix} \leqslant -\varepsilon I, \quad \forall\omega \in [-\pi, \pi], \tag{22}$$

where,

$$L_G = \begin{bmatrix} \Re(G(\exp(\mathrm{i}\omega))) & -\Im(G(\exp(\mathrm{i}\omega))) \\ \Im(G(\exp(\mathrm{i}\omega))) & \Re(G(\exp(\mathrm{i}\omega))) \end{bmatrix}. \tag{23}$$

Then the feedback interconnection is stable.

Proof. The proof follows by direct extension of Theorem 1 in [4] and is omitted. □

Remark. (i) The condition is also necessary in the sense that if Eq. (22) is not satisfied, there are signals, v, w, satisfying the QC (recall that the QC is a model for the perturbation) on Eq. (21) along with $v = Gw + f$ that violate the requirement for absolute stability (see Eq. (19)).

(ii) It is important to note that if the QCs defined by $\Pi_1(\omega), \Pi_2(\omega), \ldots, \Pi_k(\omega)$ all satisfy the perturbation class $\boldsymbol{\Delta}$ then a sufficient condition for stability is the existence of $x_1, \ldots, x_k$ such that Eq. (22) holds for

$$\Pi(\omega) = x_1 \Pi_1(\omega) + \cdots + x_k \Pi_k(\omega). \tag{24}$$

Our objective now translates to obtaining the set, $\mathscr{X}(\omega)$, of all $\Pi(\omega)$ given by

$$\mathscr{X}(\omega) = \left\{ \Pi(\omega) \in \mathbb{R}^{4n\times 4n} \;\middle|\; \begin{bmatrix} \hat{v}_r(\omega) \\ \hat{v}_i(\omega) \\ \hat{w}_r(\omega) \\ \hat{w}_i(\omega) \end{bmatrix}^{\mathrm{T}} \Pi(\omega) \begin{bmatrix} \hat{v}_r(\omega) \\ \hat{v}_i(\omega) \\ \hat{w}_r(\omega) \\ \hat{w}_i(\omega) \end{bmatrix} \geqslant 0;\; w = \Delta v;\; \forall v \in \ell_2;\; \Delta \in \boldsymbol{\Delta} \right\} \tag{25}$$

at each frequency ω such that the QC defined by them is satisfied for $w = \Delta v$, $\Delta \in \boldsymbol{\Delta}$ with $\boldsymbol{\Delta}$ defined as in Eq. (18). The motivation is to derive a finite-dimensional characterization for the set, $\mathscr{X}(\omega)$, in order to easily test the condition for stability.

It is easy to see that it suffices to find such a set for each block of perturbation. This is because if the QC defined by $\Pi_k(\omega)$ is satisfied for the kth block of perturbation then the QC given by

$$\Pi(\omega) = \mathrm{diag}[\Pi_1(\omega), \ldots, \Pi_n(\omega)] \tag{26}$$

is satisfied for $\Delta = \mathrm{diag}[\Delta_1, \ldots, \Delta_n]$. Conversely, it is easy to see that, given the diagonal block structure for the perturbation Δ every admissible $\Pi(\omega)$ will necessarily have the same diagonal structure. Therefore, we only need to consider any SISO block perturbation. With this in mind consider the following perturbation set:

$$\boldsymbol{\Delta}_0 = \{\Delta \;\mathrm{SISO} : \|\Delta\|_{\mathscr{H}(r)} \leqslant 1,\; r \in \mathscr{S}\}. \tag{27}$$

The following lemma gives a simpler convex characterization for the set $\mathscr{X}(\omega)$.

Lemma 1. *Let G and Δ be LTI stable systems with $\Delta \in \boldsymbol{\Delta}_0$. Suppose, $\mathscr{X}(\omega)$, is the set of all QCs that are satisfied for the set $\boldsymbol{\Delta}_0$. Then, verification of absolute stability for the interconnection based on the set, $\mathscr{X}(\omega)$, is equivalent to verification of absolute stability based on the set, $\mathscr{X}_0(\omega)$, given by*

$$\mathscr{X}_0(\omega) = \left\{ \Pi(\omega) \;\middle|\; \begin{bmatrix} I \\ W \end{bmatrix}^{\mathrm{T}} \Pi(\omega) \begin{bmatrix} I \\ W \end{bmatrix} \geqslant 0,\; \Pi_{11} \geqslant 0,\; \Pi_{22} \leqslant 0 \right\}. \tag{28}$$

Therefore, $\mathscr{X}(\omega) \equiv \mathscr{X}_0(\omega)$. In the future, we will make no distinction between these two sets.

Proof. For the sake of simplicity, we drop the explicit functional dependence of Π on frequency whenever it is clear from the context. Recall, from the previous section that, at every frequency ω, the image of the perturbation, Δ, is an ellipse. To this end we define

$$W(\omega) = \begin{bmatrix} \sigma_1(\omega)\cos(\theta + \psi(\omega)) & -\sigma_2(\omega)\sin(\theta + \psi(\omega)) \\ \sigma_2(\omega)\sin(\theta + \psi(\omega)) & \sigma_1(\omega)\cos(\theta + \psi(\omega)) \end{bmatrix},$$

where $\psi(\omega)$ is the orientation of the ellipse at the frequency ω, $\sigma_i(\omega)$ is the length of the minor and major axis and θ is a parameter that tracks the boundary of the ellipse as it varies from $-\pi$ to π. Note that, W, should not be confused with the weight in Eq. (7). The weight, W, introduced here represents the boundary of the set, $\Phi(\omega)$, defined in Eq. (8). With this definition the relationship between the two signals, w and u is given by

$$\begin{bmatrix} \hat{w}_r(\omega) \\ \hat{w}_i(\omega) \end{bmatrix} \in \left\{ \tau W \begin{bmatrix} \hat{v}_r(\omega) \\ \hat{v}_i(\omega) \end{bmatrix},\ \tau \in [0,1] \right\}.$$

We need τ as a variable because W alone only characterizes the outer periphery of the ellipse. Substituting the above expressions in Eq. (25) we obtain

$$\mathscr{X}(\omega) = \left\{ \Pi(\omega) \left| \begin{bmatrix} I \\ \tau W \end{bmatrix}^{\mathrm{T}} \Pi(\omega) \begin{bmatrix} I \\ \tau W \end{bmatrix} \geqslant 0,\ \tau \in [0,1] \right. \right\}.$$

In order for the set to be a useful characterization we will first need to find a means of getting rid of the dependence on τ. For this we first partition Π as follows:

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\mathrm{T}} & \Pi_{22} \end{bmatrix}.$$

We see that as we let $\tau \to 0$ we will need to have $\Pi_{11} \geqslant 0$. However, this alone is not sufficient. We claim that whenever the condition given by Eq. (22) holds it should also hold for $\tau G(\exp(\mathrm{i}\omega))$. The claim is only verified for SISO LTI discrete-time system G for simplicity.

Suppose now that the claim is not true, then there is a $v = \tau G w + f$ with (v, w) satisfying the QC defined by $\Pi(\omega)$ that violates stability in the sense of Eq. (19). But now if (v, w) satisfies the QC defined by $\Pi(\omega)$, it implies that $(v, \tau w)$ satisfies the QC too. Therefore, letting $\tilde{w} = \tau w$ we see that there is $(v, \tilde{w})$ such that $v = G\tilde{w} + f$ and $(v, \tilde{w})$ satisfies the QC. This then contradicts the stability of the interconnection of G and Δ. By substituting τG in Eq. (22) and letting τ go to zero we find that

$$\Pi_{22} \leqslant 0$$

With these substitutions we find that the set $\mathscr{X}(\omega)$ can be reduced to Eq. (28). □

Although the above expression gives a convex characterization for Π at every frequency it does not make the computation of allowable Π explicit. We next present the following theorem which gives a finite-dimensional characterization for the set of all $\Pi(\omega)$ at each frequency.

Theorem 3. *The set $\mathscr{X}(\omega)$ defined as in Eq.* (28) *has a finite-dimensional characterization. Specifically, at each frequency ω, there exist constant matrices, A, B, and linear matrix functions $C(\Pi, \omega), D(\Pi, \omega)$ in $\Pi(\omega)$ such that*

$$\Pi \in \mathscr{X}(\omega) \Leftrightarrow \begin{bmatrix} A^{\mathrm{T}}P + PA & PB - C^{\mathrm{T}}(\omega, \Pi) \\ B^{\mathrm{T}}P - C(\omega, \Pi) & -(D(\omega, \Pi) + D^{\mathrm{T}}(\omega, \Pi)) \end{bmatrix} \leqslant 0, \quad \Pi_{11} \geqslant 0, \ \Pi_{22} \leqslant 0.$$

Proof. Observing that W is a trigonometric matrix polynomial, we make a judicious transformation of variables, $\cos(\theta)$ and $\sin(\theta)$, so that we can realize the matrix trigonometric polynomial as the real-part of a finite-dimensional system. The advantage in doing this would be to apply the KYP lemma as we will see shortly. With this in mind we first partition the matrix Π_{22}:

$$\Pi_{22} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{bmatrix}$$

and re-write the expression in Eq. (28) as follows:

$$\mathscr{L}(\cos(\theta), \sin(\theta)) = \begin{bmatrix} I \\ W \end{bmatrix}^{\mathrm{T}} \Pi(\omega) \begin{bmatrix} I \\ W \end{bmatrix} \geqslant 0 \Leftrightarrow \Pi_{11} + W^{\mathrm{T}}\Pi_{12}^{\mathrm{T}} + \Pi_{12}W + W^{\mathrm{T}}\Pi_{22}W \geqslant 0. \tag{29}$$

In order to apply the KYP lemma, we transform, $\mathscr{L}(\cos(\theta), \sin(\theta))$, in terms of the variable v as follows:

$$\mathscr{L}(\cos(\theta), \sin(\theta)) = \Re(D + C(\mathrm{j}v - A)^{-1}B).$$

The first term of the last expression in Eq. (29) is a constant. W which enters the second and third terms can be realized as the real-part of a finite-dimensional system by making the following substitutions:

$$\cos(\theta)=\frac{1-v^2}{1+v^2},\quad \sin(\theta)=\frac{2v^2}{1+v^2},\quad -\infty\leqslant v\leqslant\infty$$

and obtain

$$W=\Re\left(\begin{bmatrix} a_{11}(\omega)\frac{1-jv}{1+jv}+b_{11}(\omega)\frac{jv}{1+jv} & -a_{12}(\omega)\frac{1-jv}{1+jv}-b_{12}(\omega)\frac{jv}{1+jv} \\ a_{12}(\omega)\frac{1-jv}{1+jv}+b_{12}(\omega)\frac{jv}{1+jv} & -a_{11}(\omega)\frac{1-jv}{1+jv}+b_{11}(\omega)\frac{jv}{1+jv}\end{bmatrix}\right).$$

The finite-dimensional system corresponding to the factors of W are of the form $(1-s)/(1+s)$ and $s/(1+s)$. We may therefore write the second and third terms of the last expression in Eq. (29) as

$$W^{\mathrm{T}}\Pi_{12}^{\mathrm{T}}+\Pi_{12}W=\frac{\Pi_{12}L_0(\omega)+L_0^{\mathrm{T}}(\omega)\Pi_{12}^{\mathrm{T}}}{(s+1)}+\Pi_{12}M_0(\omega)+M_0^{\mathrm{T}}(\omega)\Pi_{12}$$

which is easy to transform to state-space realization. The last term of Eq. (29) can be rewritten as

$$\sigma_1(\omega)\cos(\theta+\psi(\omega))W\Pi_{22}+\sigma_1(\omega)\sin(\theta+\psi(\omega))\Pi_{22}\begin{bmatrix}0&1\\-1&0\end{bmatrix}.$$

Thus we need to express the terms $\cos^2(\theta),\sin^2(\theta),\sin(\theta)\cos(\theta)$ as finite-dimensional systems given our choice for $\cos(\theta),\sin(\theta)$. This turns out to be fairly easy as seen below:

$$\cos^2(\theta)=1-\sin^2(\theta)=\frac{1}{2}\left(\Re\left(\frac{1-jv}{1+jv}\right)^2+1\right),$$

$$\sin(\theta)\cos(\theta)=\Re\left(\frac{(2jv(1-jv)}{(1+jv)^2}+\frac{1}{2}\left(1-\frac{(1-jv)^2}{(1-jv)^2}\right)\right).$$

With these expressions the matrices, A,B,C,D in Eq. (29) can be computed. As one can readily see the matrices A and B are constant matrices that do not depend on Π or on the frequency ω. The matrices C and D are given by the following expressions:

$$D(\omega,\Pi)=\Pi_{11}+\Pi_{12}D_0(\omega)+D_0^{\mathrm{T}}(\omega)\Pi_{12}^{\mathrm{T}}+E_1(\omega)\Pi_{22}D_1(\omega)+E_2(\omega)\Pi_{22}D_1,$$

$$C(\omega,\Pi)=\Pi_{12}C_0(\omega)+C_0^{\mathrm{T}}(\omega)\Pi_{12}^{\mathrm{T}}+E_1\Pi_{22}C_1(\omega)+E_1(\omega)\Pi_{22}C_1,$$

where C_i,D_i,E_i are all constants depending only on the frequency ω. Now by applying the KYP (see [9]) lemma we obtain that the above expression is equivalent to an LMI, i.e.,

$$\Re\left(D(\omega,\Pi)+C(\omega,\Pi)(jv-A)^{-1}B(\omega)\right)\geqslant 0\Leftrightarrow\begin{bmatrix}A^{\mathrm{T}}P+PA & PB-C^{\mathrm{T}}(\omega,\Pi)\\ B^{\mathrm{T}}P-C(\omega,\Pi) & -(D(\omega,\Pi)+D^{\mathrm{T}}(\omega,\Pi))\end{bmatrix}\leqslant 0. \quad (30)$$

Thus the set $\mathscr{X}(\omega)$ can be reduced to the feasibility of a finite-dimensional LMI. □

It is now easy to provide a finite-dimensional expression for $\mathscr{X}(\omega)$ for the spatially structured perturbation as in Eq. (18). We present this here for the sake of completion. Corresponding to each diagonal perturbation element. Δ_k, in Eq. (18), there is a corresponding set, $\mathscr{X}_k(\omega)$ that can be re-written by means of a finite-dimensional LMI. Thus, the set $\mathscr{X}(\omega)$, can be expressed as

$$\mathscr{X}(\omega)=\{\mathrm{diag}[\Pi_1(\omega),\Pi_2(\omega),\ldots,\Pi_n(\omega)]\colon \Pi_j(\omega)\in\mathscr{X}_j(\omega),\ j=1,2,\ldots,n\}.$$

We know from Theorem 3 that every set $\mathscr{X}_k(\omega)$ can be re-written as an LMI defined by matrices A_k,B_k, $C_k(\omega,\Pi_k)$, $D_k(\omega,\Pi_k)$ for each frequency $\omega\in[-\pi,\pi]$. Therefore, the above condition for $\mathscr{X}(\omega)$ can be expressed as

$$\Pi(\omega)\in\mathscr{X}(\omega)\Leftrightarrow\begin{bmatrix}A_k^{\mathrm{T}}P_k+P_kA_k & P_kB_k-C_k^{\mathrm{T}}(\omega)\\ B_k^{\mathrm{T}}P_k-C_k(\omega,\Pi_k) & -(D_k(\omega,\Pi_k)+D_k^{\mathrm{T}}(\omega,\Pi_k))\end{bmatrix}\leqslant 0,\quad \Pi_{k11}\geqslant 0,\Pi_{k22}\leqslant 0,$$
$$k=1,2,\ldots,n, \quad (31)$$

where $\Pi(\omega)=\mathrm{diag}[\Pi_1(\omega),\ldots,\Pi_n(\omega)]$. This in turn is a finite-dimensional LMI feasibility condition.

4.1. Robustness of real-parameter uncertainty

In this section we derive sufficient conditions for real-parametric uncertainty by embedding the uncertainty in a Hilbert space. In this way the techniques developed in the previous section can form the basis for robust stability for mixed real-parametric and non-parametric uncertainty. Thus we are left to find an appropriate Hilbert space (in particular a weighting sequence, $r(k)$) that approximates the real-parameter uncertainty. To this end, let $v=\gamma w$ where $\gamma \in [-1,1]$. Then,

$$[-1,1] \subset \left\{ \Delta \,\middle|\, \sum_{k=0}^{\infty} (\delta(k)\rho^k)^2 \leqslant 1,\ \rho \in (1,\infty] \right\}.$$

As we let $\rho \to \infty$ we get increasingly better approximations to the real-parameter γ. Now, stability for perturbations belonging to a Hilbert-space are sufficient conditions for stability of real-parametric uncertainties. By applying Theorems 2 and 3 we can reduce the problem to a finite-dimensional LMI condition. Moreover, by incrementing the parameter ρ we obtain less-conservative conditions for stability for real-parametric uncertainties.

5. Conclusions

In this paper we have obtained a convex parameterization of robustly stabilizing controllers for SISO perturbations that belong to Hilbert spaces. These methods can be extended to SIMO or MISO block perturbations with mixed-parametric and non-parametric parts. We have also derived robustness analysis tests based on the $\mathcal{S}$ procedure. We have shown that the verification of robust stability can be reduced to the verification of a finite-dimensional linear matrix inequality (LMI). These problems are naturally motivated from a system-identification perspective. System identification of infinite-dimensional systems belonging to a Hilbert space with finite noisy data results in estimates of nominal model along with non-parametric and parametric error. The non-parametric error is typically described by a unit ball in the infinite-dimensional Hilbert space. Such uncertain models form the basis for pursuing control techniques and leads to a systematic methodology for going from data to control.

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