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SIGNAL RECONSTRUCTION UNDER FINITE-RATE MEASUREMENTS: FINITE-HORIZON NAVIGATION APPLICATION

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ABSTRACT

In this paper, we study finite-length signal reconstruction over a finite-rate noiseless channel. We allow the class of signals to belong to a bounded ellipsoid and derive a universal lower bound on a *worst-case* reconstruction error. We then compute upper bounds on the error that arise from different coding schemes and under different causality assumptions. We then map our general reconstruction problem into an important control problem in which the plant and controller are local to each other, but are together driven by a remote reference signal that is transmitted through a finite-rate noiseless channel. The problem is to navigate the state of the remote system from a nonzero initial condition to as close to the origin as possible in finite-time. Our analysis enables us to quantify the tradeoff between time horizon and performance accuracy which is not well-studied in the area of control with limited information as most works address infinite-horizon control objectives (eg. stability, disturbance rejection).

I. INTRODUCTION

Signal reconstruction over noisy channels has been well studied under stochastic settings, where performance criteria are typically characterized by asymptotic properties of the probability of error given *stochastic* descriptions of the input signal and channel. The main objective of signal reconstruction is to design computationally efficient coding schemes to optimize performance [14], [20]. Recent work by Voulgaris investigates reconstruction of *infinite-length discrete-valued* signals that are filtered via noisy channels using a *deterministic* framework [24]. In contrast, we study *finite-length real-valued* signal reconstruction filtered via finite-rate but otherwise noiseless channels using a deterministic framework. In particular, we are interested in minimizing reconstruction error in *finite-time*, whereas in most communication settings questions about *asymptotic* reconstruction are typically addressed. We study finite-time performance because we are ultimately interested in understanding how the reconstructed signal can be used to drive or control a system.

Control over noisy channels is a research area of growing interest. Today new problems in control over networked systems, whose components are connected via communication links that can be very noisy, induce delays, and have finite rate constraints, are emerging. Applications include remote navigation systems (deep-space and sea exploration) and multi-robot control systems (eg. aircraft and spacecraft formation flying control, coordinated control of land robots, control of multiple surface and underwater vehicles), where robots exchange data through communication channels that impose constraints on the design of coordination strategies.

Much work in the area of control with limited information has focused on stability under finite-rate (or countable) feedback control, where the only excitation to the system is an unknown (but bounded) initial state condition [2], [3], [4], [5], [8], [15], [19], [21], [23]. The questions posed involve conditions on the channel rate that will guarantee that the state of the system (or some function of the state) approach the origin/remain bounded as time goes to infinity. More recently, disturbance rejection limitations were derived for the same setting, assuming stochastic exogenous signals entering the system [17], [18]. Although these studies greatly contribute to our understanding of the interplay between communication and control, few studies have addressed finite-horizon performance limitations under communication constraints.

A handful of recent studies explore the tradeoffs between finite-horizon performance and control complexity for linear systems and finite automata systems [6], [7], [9], [10]. A navigation problem similar in spirit to that which is presented here is described in [10]. In this paper we introduce and analyze a general signal reconstruction framework which enables us to compute a universal lower bound for finite-horizon navigation under finite-rate *feedforward* control. That is, we compute the smallest allowable ball around the origin that the state of the system can reach in T time steps under finite-rate constraints, given that the initial condition lies in an ellipsoid. We also construct two quantization/coding schemes to derive upper bounds on our performance metric.

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II. GENERAL RECONSTRUCTION PROBLEM

In this section, we define a framework to study finite-length signal reconstruction under finite-rate measurements. We consider the cascade of SISO discrete-time systems shown in Figure 1.

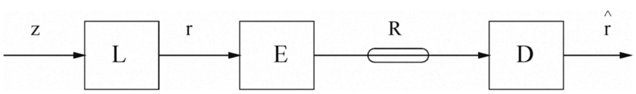

Fig. 1. General Reconstruction Set Up

Specifically,

- $z \in \mathbb{R}^n$ $s.t.$ $||z||_2 \leq 1$,
- $r \in \mathcal{C}_r \triangleq \{r \in \mathbb{R}^n, z \in \mathbb{R}^n | r = Lz, \; ||z||_2 \leq 1\}$,
- $L : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear operator,
- $E : \mathbb{R}^n \to \{0,1\}^{RT}$ is an arbitrary operator (encoder) that maps a real vector to a sequence of 2^{RT} binary symbols where $T \geq n$,
- R is the channel rate for the finite-rate noiseless channel that maps $\{0,1\}^{RT} \to \{0,1\}^{RT}$, and
- $D : \{0,1\}^{RT} \to \mathbb{R}^n$ is an arbitrary operator (decoder) that maps a sequence of 2^{RT} binary symbols to a real vector.

Note that L defines a class of finite-length signals, $\mathcal{C}_r$, that is generated from a unit ball in $\mathbb{R}^n$. Since L is linear, it maps the unit ball to a bounded ellipsoid in $\mathbb{R}^n$. We assume L and the channel rate R are given, and we want to find an encoder E and decoder D to minimize a reconstruction error over all signals, r, in this class (worst-case analysis). Since the input and output signals have finite length, the following performance metric: $||W(r - \hat{r})||_2^2$, where $W \in \mathbb{R}^n \times \mathbb{R}^n$ is a given full-rank weight matrix.

To understand reconstruction limitations under finite-rate measurements, we compute γ_{LB} and γ_{UB}, such that

$$\gamma_{LB} \leq \min_{(E,D)} \; \sup_{r \in \mathcal{C}_r} ||W(r - \hat{r})||_2^2 \leq \gamma_{UB}.$$

Knowledge of γ_{LB} tells us that regardless of the encoder and decoder that we select, we can do no better than this universal lower bound. The upper bound tells us that there exists a coding scheme such that the worst case performance is always less than or equal to γ_{UB}. To compute γ_{UB}, we construct an encoder and decoder and compute the corresponding worst-case performance. In the following sections, we compute a universal lower bound and two upper bounds corresponding to two types of coding schemes.

III. UNIVERSAL LOWER BOUND

In this section we state a universal lower bound on worst-case reconstruction which can be proved using a standard counting or sphere-packing type argument (proof is given in [22]) and can also from a proof given in [23].

Theorem III.1. *Given the signal reconstruction set up defined above, assume that $det(W) \neq 0$, $det(L) \neq 0$. Then,*

$$\gamma_{LB} = 2^{-2RT/n} \{|det(L)| \; |det(W)|\}^{\frac{2}{n}}.$$

When computing the lower bound, we made no assumptions on whether the encoder and decoder are causal or noncausal operators. If both the encoder and decoder are noncausal, then at time $t = 0$ the decoder "knows" the future. That is, at $t = 0$ it can compute $\hat{r}_k$ for $k = 0, 1, ..., n-1$ which are represented by TR bits over a horizon of T steps and our reconstruction problem reduces to a vector quantization (VQ) problem with a deterministic error metric [1], [12]. If we consider our signal r to be a correlated Gaussian random vector with covariance matrix M_r, then the lower bound on the minimum mean-squared error ($E(r-\hat{r})^T(r-\hat{r})$) for $T = n$ is $2^{-2R} det(M_r)^{\frac{1}{n}}$ [13], which is identical to our lower bound if we replace WL with $M_r^{\frac{1}{2}}$. This makes sense as the mean-squared error lower bound is for every possible realization $\hat{r}$ and hence considers the worst case.

We still, however, find it useful to derive γ_{LB} in our deterministic setting and compare it to upper bounds computed under various coding schemes such as when decoding must be done in a causal manner. Recall that if If E and D are causal, then at time k the encoder can only operate on $r_0, r_1, ..., r_k$ and decoder can only reconstruct $\hat{r}_0, ..., \hat{r}_k$ where $\hat{r}_k$ is represented by at most $(k+1)R$ bits. In the following sections, we compute two upper bounds. The first bound is computed by constructing a noncausal encoder and decoder (γ_{NN}), and the second is computed by constructing a noncausal encoder and a causal decoder (γ_{NC}). In [22] we construct a causal coding scheme and compute (γ_{CC}) and show how imposing causality imposes severe performance constraints.

IV. NONCAUSAL ENCODING AND DECODING

In this section, we derive an upper bound, γ_{NN}, on worst-case performance assuming that the encoder and decoder are both noncausal. The upper bound is computed using a coding scheme that transmits information about the signal r in terms of a basis derived from the singular value decomposition (SVD) of the matrix WL. This scheme is identical to that proposed for a Gaussian source signal in [13], however in [13] the corresponding upper bound was not computed exactly. Whereas, here we compute γ_{NN} exactly for this scheme.

Consider Figure 2. Let the SVD decomposition of WL be defined as $WL = U\Sigma V^*$, where U is an $n \times n$ unitary matrix, Σ is an $n \times n$ diagonal matrix containing the singular values of WL, and V is a $n \times n$ unitary matrix. The encoder consists of rotator and quantizer operators, p and q, respectively which are defined as follows:

- $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $p(r) = V^* Lr$,

- $q : \mathbb{R}^n \rightarrow \{0,1\}^{RT}$ is parameterized by the bit-allocation strategy denoted in $\mathcal{R} = (R_0, R_1, ..., R_{n-1})$ as shown in Figure 2 where $\hat{\alpha}_i = q_i(\alpha_i)$ and $q_i(\alpha_i)$ is a uniform quantizer from [-1, 1] with 2^{R_i} levels for $i = 0, 1, ..., n-1$ ($\alpha = (\alpha_0 \ \alpha_1 \ ... \ \alpha_{n-1})$.

The decoder first employs the bit-allocation strategy $\mathcal{R}$ to reconstruct $\hat{\alpha}$ from the binary string that it receives. It then rotates the vector to compute $\hat{r}$. Specifically, $\hat{r} = L^{-1}V\hat{\alpha}$. We call this $E-D$ construction the "SVD Coding Scheme."

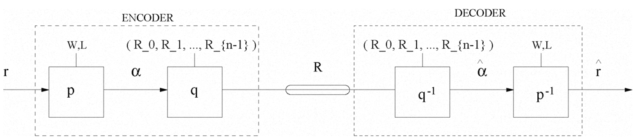

Fig. 2. SVD Coding Scheme

Note that for the above SVD coding scheme,

$$
\begin{aligned}
\sup_{r \in \mathcal{C}_r} ||W(r-\hat{r})||_2^2 &= \sup_{\{z| \ ||z||_2 \le 1\}} ||WL(\hat{z}-z)||_2^2 \\
&= \sup_{\{z| \ ||z||_2 \le 1\}} ||U\Sigma V^*(\hat{z}-z)||_2^2 \\
&= \sup_{\{\alpha| \ ||\alpha||_2 \le 1\}} ||\Sigma(\hat{\alpha}-\alpha)||_2^2 && (1) \\
&= \sup_{\{\alpha| \ ||\alpha|_2 \le 1\}} \sum_{i=0}^{n-1} (\hat{\alpha}_i - \alpha_i)^2 \sigma_i^2 && (2) \\
&\le \sup_{\{\alpha| \ ||\alpha|_2 \le 1\}} \sum_{i=0}^{n-1} 2^{-2R_i} \sigma_i^2 |\alpha_i|^2 && (3) \\
&\le \max_i 2^{-2R_i} \sigma_i^2 \sup_{\{\alpha| \ ||\alpha|_2 \le 1\}} \sum_{i=0}^{n-1} |\alpha_i|^2 && (4) \\
&= \max_i 2^{-2R_i} \sigma_i^2. && (5)
\end{aligned}
$$

To derive the upper bound γ_{NN} using the above SVD coding scheme, we construct $\mathcal{R} = (R_0, R_1, ..., R_{n-1})$ to solve the following optimization problem:

$$
\begin{aligned}
\min_{\mathcal{R}} \quad & \max_i 2^{-2R_i} \sigma_i^2 && (6) \\
s.t. \quad & \textstyle\sum_{i=0}^{n-1} R_i \le TR \\
& R_i \ge 0 \ \ \forall i.
\end{aligned}
$$

Problem (6) is equivalent to the following optimization problem:

$$
\begin{aligned}
\min_{\mathcal{R},\gamma} \quad & \gamma && (7) \\
& 2^{-2R_i}\sigma_i^2 \le \gamma \\
s.t. \quad & \textstyle\sum_{i=0}^{n-1} R_i \le TR \\
& R_i \ge 0 \ \ \forall i. \\
& \gamma \ge 0.
\end{aligned}
$$

[1] Now note that a lower bound to the optimal cost of (7) is the optimal solution to

$$
\begin{aligned}
\min_{\mathcal{R},\gamma} \quad & \gamma + \textstyle\sum_{i=0}^{n-1} \lambda_i(-2R_i + 2log(\sigma_i) - log(\gamma)) + && (8) \\
& \mu(\textstyle\sum_{i=0}^{n-1} R_i - TR) \\
& R_i \ge 0 \ \ \forall i. \\
& \gamma \ge 0,
\end{aligned}
$$

where $\mu \ge 0$ and $\lambda_i \ge 0$ for all i. We rearrange terms to get

$$
\begin{aligned}
\min_{\gamma \ge 0} \quad & [\gamma - \textstyle\sum_{i=0}^{n-1} \lambda_i log(\gamma)] + && (9) \\
& \min_{R_i \ge 0} [\textstyle\sum_{i=0}^{n-1} R_i(\mu - 2\lambda_i)] + \\
& [2\textstyle\sum_{i=0}^{n-1} \lambda_i log(\sigma_i) - \mu RT].
\end{aligned}
$$

The minimization over nonnegative R_i (second term in (9)) is as follows

$$
R_i^* = \begin{cases} 0 & \mu - 2\lambda_i \ge 0 \ \ \forall i \\ -\infty & o.w. \end{cases}
$$

which gives us

$$
\begin{aligned}
\min_{\gamma \ge 0} \quad & \gamma - \textstyle\sum_{i=0}^{n-1} \lambda_i log(\gamma) + 2\textstyle\sum_{i=0}^{n-1} \lambda_i log(\sigma_i) - \mu RT. && (10) \\
s.t. \quad & 0 \le \lambda_i \le \tfrac{\mu}{2} \ \ \forall i
\end{aligned}
$$

since we know a finite solution to (7) exists. Now, if we minimize over γ we get

$$
\gamma^* = \sum_i \frac{\lambda_i^*}{ln(2)}, \tag{11}
$$

[1] We allow the rates to take on non-integer values to solve for an optimal bit-allocation strategy. The resulting non-integer valued rates can be interpreted as average rates over time.

and the dual to (7) is

$$\begin{array}{lc} \max\limits_{\lambda_0,\lambda_1,...,\lambda_{n-1},\mu} & \sum_i \lambda_i(\frac{1}{ln(2)} - log(\sum_i \frac{\lambda_i}{ln(2)})) + \quad (12) \\ & \sum_i 2\lambda_i log(\sigma_i) - \mu RT \\ s.t. & 0 \leq \lambda_i \leq \frac{\mu}{2} \quad \forall i \\ & \mu \geq 0. \end{array}$$

It is fairly straightforward to show that the cost to (12) is a convex function of $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ over a bounded set, and therefore the optimal solution occurs at a boundary. That is,

$$\lambda_i^* = \begin{cases} \frac{\mu}{2} & i \in I \\ 0 & i \in I^c, \end{cases} \tag{13}$$

where $I \subset \{0, 1, ..., n-1\}$. Plugging (13) into (12), the dual becomes

$$\begin{array}{lc} \max\limits_{\mu} & \frac{\mu|I|}{2}(\frac{1}{ln(2)} - log(\frac{\mu|I|}{2ln(2)}) + \quad (14) \\ & \mu \sum_{i\in I} log(\sigma_i) - \mu RT \\ & \mu \geq 0. \end{array}$$

One can compute the solution to (14) as

$$\mu^* = \frac{2ln(2)}{|I|} 2^{\frac{-2RT}{|I|}} \prod_{i\in I} \sigma_i^{\frac{2}{|I|}}.$$

Finally, we plug μ^* into (13) and then (11) to get

$$\gamma^* = \gamma_{NN} = \max \, \{2^{\frac{-2RT}{|I|}} \prod_{i\in I} \sigma_i^{\frac{2}{|I|}}, \max_{i\in I^c} \; \sigma_i^2\}. \tag{15}$$

One can show that the dual cost equals the primal cost when $R_i^* =$

$$\begin{cases} log(\sigma_i) + \frac{RT}{|I|} - \frac{1}{|I|}\sum_{i\in I} log(\sigma_i) & for \;\; i \in I \\ 0 & o.w. \end{cases}$$

Since $\lambda_i^* = 0$ for all $i \in I^c$, necessarily $R_i^* = 0$ for $i \in I^c$ since the 2nd term in (4) must equal 0. As expected, the optimal rate allocation places more bits to components of α whose corresponding singular values are larger. Surprisingly, if $|I| = n$, then $R_i^* = R + log(\sigma_i) - \frac{1}{n}\sum_{j=0}^{n-1} log(\sigma_j)$ for $i = 0, 1, ..., n-1$ and the resulting upper bound is $\gamma_{NN} = 2^{-2RT/n}\{\prod_{i=0}^{n-1}\sigma_j\}^{\frac{2}{n}} = \gamma_{LB}$.

Finally, we comment on the construction of I that minimizes γ^*. Note that for the rates R_i to all be nonnegative, we require that

$$log(\sigma_i) \geq \frac{1}{|I|}\sum_{i\in I} log(\sigma_i) - \frac{RT}{|I|} \quad \forall i \in I \tag{16}$$

$$\Rightarrow \qquad \sigma_i^2 \geq 2^{\frac{-2RT}{|I|}} \prod_{i\in I} \sigma_i^{\frac{2}{|I|}} \quad \forall i \in I.$$

Comparing the above inequality to the expression for γ^*, we see that in order to minimize γ^*, we want to place all the indices corresponding to the larger singular values in the index set I until the positivity constraints on the rates are violated. Note that this scheme is reminiscent of a water-filling problem in coding.

V. NONCAUSAL ENCODING AND CAUSAL DECODING

In this section, we derive an upper bound, γ_{NC}, by constructing a modified SVD Coding Scheme in which the encoder is noncausal, *i.e.,* has access to the entire signal $r \in \mathcal{C}_r$ at time $t = 0$, but the decoder is causal. The scheme we propose, sketched in Figure 3, is similar to that described in section IV with the restriction that the decoder can only process R bits of information at each time step. As a result, the rotator operator p remains unchanged from that defined in section IV ($p : \mathbb{R}^n \to \mathbb{R}^n$ such that $p(r) = V^*Lr$, where $WL = U\Sigma V^*$), but the quantizer operator q changes.

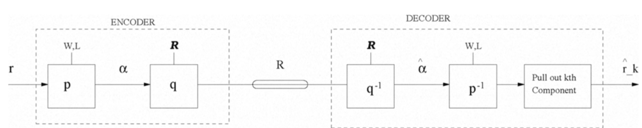

Fig. 3. SVD Scheme for Noncausal Encoding and Causal Decoding

At each time step t, the quantizer has R bits that it allocates to the entire vector α. The bit-allocation is determined by the following rate matrix

$$\mathcal{R}_{NC} = \begin{bmatrix} R_{01} & R_{12} & & \dots & R_{0,n-1} \\ R_{11} & R_{12} & & & R_{1,n-1} \\ R_{21} & R_{22} & & & R_{2,n-1} \\ \vdots & \vdots & & & \vdots \\ R_{T-1,1} & R_{T-1,2} & R_{T-1,3} & \dots & R_{T-1,n-1} \end{bmatrix},$$

such that $\sum_j R_{ij} = R$ for $i = 0, 1, ..., T-1$. More specifically, let $R_i(t) = \sum_{j=0}^{t} R_{ji}$ for $i = 0, 1, ..., n-1$ and $t = 0, 1, ..., T-1$. Then, at time t, a total of $R_i(t)$ bits are allocated to α_i to produce $\hat{\alpha}_i(t)$ for $i = 0, 1, ..., n-1$. The decoder then produces an estimate of the entire signal $\hat{r}(t) = L^{-1}V\hat{\alpha}(t)$ and pulls out the kth component. Note that $\hat{r}(t) = (\hat{r}_0(t), \hat{r}_1(t), ..., \hat{r}_{n-1}(t))$ and one can show that the worst-case cost is then

$$\sup_{r\in\mathcal{C}_r} ||W(r-\hat{r})||_2^2 \leq \max_i 2^{-2R_i(i)}\sigma_i^2.$$

To derive the upper bound γ_{NC} using the above SVD coding scheme, we construct rate matrix $\mathcal{R}_{NC}$ to solve the following optimization problem:

$$\begin{aligned} \min_{\mathcal{R}_{NC}} \quad & \max_i 2^{-2R_i(i)}\sigma_i^2 \\ s.t. \quad & \textstyle\sum_{j=1}^{T-1} R_{ij} = R \quad for \quad i = 0, 1, ..., n-1 \\ & R_{ij} \geq 0 \quad \forall i, j. \end{aligned} \tag{17}$$

Note that the above optimization problem is similar to (6) which computes γ_{NN}. We note that there may exist singular values σ_i for $i = 0, 1, ..., n-1$ and channel rates R that may result in $\gamma_{NC} = \gamma_{NN}$. Next, we illustrate a practical navigation control problem that gives rise to a noncausal encoder and causal decoder pair.

VI. FINITE-HORIZON CONTROL APPLICATION

In this section, we show how the above analysis (in particular when the encoder is noncausal and the decoder is causal) of general finite-length signal reconstruction enables us to quantify bounds on performance for finite-horizon finite-rate navigation. Finite-horizon finite-rate tracking is discussed in [22].

Assume that the remote system has some unknown initial condition x_0 which lies in a known bounded ellipsoid in $\mathbb{R}^n$ and we want to steer the state of the remote system as close to the origin as possible under the constraint that the control input can take on at most 2^{RT} values after T time steps, *i.e.,* the command is transmitted through a finite-rate noiseless channel. This navigation problem can be analyzed as the cascade of systems shown in Figure 4.

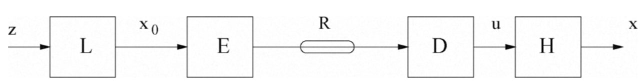

Fig. 4. Finite Horizon Navigation Set Up

Specifically,

- $z \in \mathbb{R}^n$ $s.t.$ $||z||_2 \leq 1$,
- $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator,
- $E : \mathbb{R}^n \to \{0,1\}^{RT}$ is an arbitrary operator (encoder) that maps a real vector to a sequence of 2^{RT} binary symbols,
- R is the channel rate for the finite-rate noiseless channel that maps $\{0,1\}^{RT} \to \{0,1\}^{RT}$,
- $D : \{0,1\}^{RT} \to \mathbb{R}^T$ is an arbitrary operator (decoder) that maps a sequence of 2^{RT} binary symbols to a real vector, and
- H is a causal SISO LTI system with state-space representation $H = ss(A, B, I, 0)$ with (A, B) reachable and A is full rank; and the state vector at time t is x_t.

Our navigation metric is $\min_{(E,D)} \sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2^2$ and we next compute a universal lower bound for it in a similar fashion as that computed in section III. First note that $x_T = A^T x_0 + Mu$, where $Mu = \sum_{i=0}^{T-1} A^{-(i+1)} Bu_i$. Since the system H is reachable, M is a $n \times T$ full rank matrix, and if we know x_0 exactly then we can construct a control input u such that $Mu = -A^T x_0$ which would make $x_T = 0$. Under finite-rate constraints, we must construct u such that $Mu = -A^T \hat{x}_0$, where $\hat{x}_0$ can take on at most 2^{RT} values in $\mathcal{C}_{x_0}$. The corresponding metric is then $\min_{(E,D)} \sup_{x_0 \in \mathcal{C}_{x_0}} ||A^T(x_0 - \hat{x}_0)||_2^2$, which looks like our general reconstruction error metric $\min_{(E,D)} \sup_{r \in \mathcal{C}_r} ||W(r - \hat{r})||_2^2$ with W replaced with A^T and r replaced with x_0.

Now, we show how we implement the coding scheme described in section V to compute an upper bound,γ_{NC}^{NAV}. Specifically, we consider the following navigation error

$$\min_{\mathcal{R}_{NC}, u} \sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2^2. \tag{18}$$

where $\mathcal{R}_{NC}$ is a rate matrix introduced in section V. Note that $x_T = A^T x_0 + Mu = A^T(x_0 - \hat{x}_0) + Mu + A^T \hat{x}_0$, where again $Mu = \sum_{i=0}^{T-1} A^{-(i+1)} Bu_i$. Therefore,

$$\begin{aligned} & \min_{\mathcal{R}_{NC},\ u} \sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2 = \\ & \min_{\mathcal{R}_{NC}, u} \sup_{x_0 \in \mathcal{C}_{x_0}} ||A^T(x_0 - \hat{x}_0) + Mu + A^T \hat{x}_0||_2 \\ & \leq \min_{\mathcal{R}_{NC}} \sup_{x_0 \in \mathcal{C}_{x_0}} ||A^T(x_0 - \hat{x}_0)||_2 + \\ & \qquad \min_u ||Mu + A^T \hat{x}_0||_2. \end{aligned} \tag{19}$$

The two terms in the last inequality above is computed by first applying the coding scheme presented in section V to compute the $T \times n$ rate matrix, $\mathcal{R}_{NC}^*$, that minimizes $\sup_{x_0 \in \mathcal{C}_{x_0}} ||A^T(x_0 - \hat{x}_0)||_2$, and then computing the control input u that minimizes $||Mu + A^T \hat{x}_0||_2$ given $\mathcal{R}_{NC}^*$. This would give the following upper bound on the navigation error (18)

$$\begin{aligned} & \min_{\mathcal{R}_{NC}, u} \sup_{x_0 \in \mathcal{C}_{x_0}} ||x_T||_2^2 \leq \\ & \gamma_{NC} + \min_u ||Mu + A^T \hat{x}_0||_2, \end{aligned}$$

where γ_{NC} is the optimal cost to (17), where σ_i is the i'th singular value of $A^T L$. Finally, $\min_u ||Mu + A^T \hat{x}_0||_2$ is solved assuming the causal decoder computes the control input u_t for each time step t by solving the following optimization problem for $t \geq 0$:

$$\min_{u_t} \quad \sup_{\alpha \in S_t} ||AU\Sigma\alpha + \textstyle\sum_{i=0}^{t-2} A^{t-1-i} Bu_i + Bu_t||_2 \tag{20}$$

where $S_t = \{\alpha \in \mathbb{R}^n |\ |\alpha_i - \hat{\alpha}_i(t)| \leq |\alpha_i| 2^{-R_i(t)} \quad i = 1, 2, ..., n\}$. The solution to (20) can be computed in a straightforward manner and is $u_t^* = \frac{(AU\Sigma\alpha^*(t) + \sum_{i=0}^{t-2} A^{t-1-i} Bu_i)' B}{(B'B)}$.

VII. PERFORMANCE COMPARISON

In this section, we fix H to be an LTI system and quantify tradeoffs between time horizon and performance navigation. Performance analysis for finite horzon tracking can be found in [22]. We compare the lower and upper navigation bounds on γ to each other for a given causal LTI stable system $H = ss(A, B, C, D)$, for different initial condition ellipsoids (L) and time horizons T. We consider diagonal 4×4 ($n = 4$) state-transition matrix $A = diag(0.2, 0.8, 0.9, 0.8)$, two L matrices that are generated by LTI system $ss(A_l, B_l, C_l, D_l)$, and we fix the rate $R = 5$.

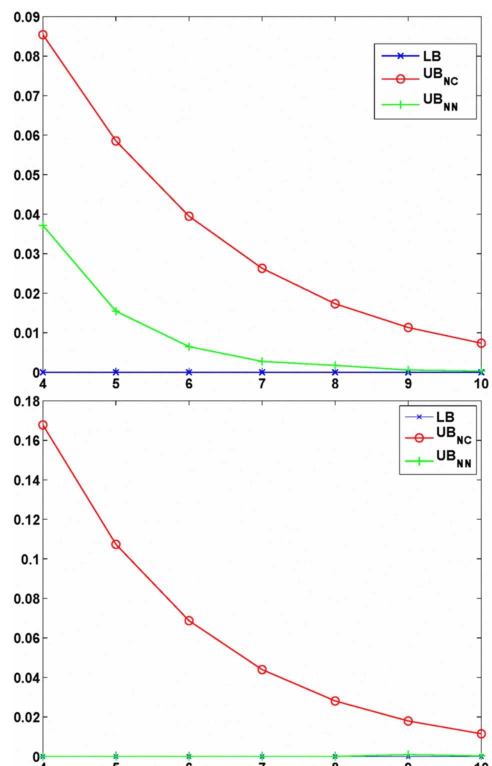

Fig. 5. Top: Bounds for $L = ss(0.99, 0.99, 1, 1)$ Bottom: Bounds for $L = ss(0.01, 0.01, 1, 1)$

Figure 5 illustrates the bounds for different system parameters from which we make the following observations. All bounds decay when A is stable as T grows. When the pole of the system that generates L is closer to the origin, the singular values of L are all comparable and therefore using the SVD basis to represent x_0 in the causal coding scheme is less helpful. Therefore, we expect causal coding performance to deteriorate, which it does. Put another way, when L has a pole close to the unit disk, then the ellipsoid set $\mathcal{C}_{x_0}$ has more structure, that is knowing some components of x_0 give a lot of information about the remaining component of x_0. When the pole of L is closer to 0, then $\mathcal{C}_{x_0}$ looks more and more like an n-dimensional sphere.

VIII. REFERENCES

[1] Allen, G, Gray, RM. *Vector Quantization ans Signal Compression.* Kluwer Academic Publishers, 1992.

[2] Baillieul, J. Feedback Coding for Information-based Control: Operating Near the Data-Rate Limit. *Proceedings of the 41st IEEE Conference on Decision and Control* 2002.

[3] Wong, WS, Brockett, R. Systems with Finite Communication Bandwidth Constraints-I: State Estimation Problems. *IEEE Transactions on Automatic Control* 1997; **42**(9).

[4] Brockett, R. Quantized Feedback Stabilization of Linear Systems. *IEEE Transactions on Automatic Control* 2000; **45**: 1279-1289.

[5] Wong, WS, Brockett, R. Systems with Finite Communication Bandwidth Constraints-II: Stabilization with Limited Information Feedback. *IEEE Transactions on Automatic Control* 1999; **44**(5).

[6] Delevenne, JC, Blondel, V. Complexity of Control of Finite Automata. *IEEE Transactions on Automatic Control* 2006; **51**(6): 977- 986.

[7] Delevenne, JC. An optimal quantized feedback strategy for scalar linear systems. *IEEE Transactions on Automatic Control* 2006; **51**(2): 298-303.

[8] Elia, N, Mitter, S., Stabilization of Linear Systems With Limited Information *IEEE Transactions on Automatic Control* 2001; **46**(9).

[9] Fagnani, F, Zampieri, S. Stability Analysis and Synthesis for Linear Systems With Quantized Feedback. *IEEE Transactions on Automatic Control* 2003; **48**(9).

[10] Fagnani, F, Zampieri, S. Quantized Stabilization of Linear Systems: Complexity Versus Performance. *IEEE Transactions on Automatic Control* 2004; **49**(9).

[11] Gallager, R., *Information Theory and Reliable Communication.* John Wiley and Sons, Inc.: New York, 1968.

[12] Gersho, A., Gray R.M., *Vector Quantization and Signal Compression* Kluwer Academic Publishers: Boston, c1992.

[13] Huang J.J.Y., Schultheiss, P.M., Block Quantization of Correlated Gaussian Random Variables *IEEE Transactions on Communication System* 1963; **11**(3), 289-296.

[14] Lee, EA, Messerschmitt, DG. *Digital Communication.* (2nd edn) Kluwer Academic Publishers, 1994.

[15] Liberzon, D., A note on stabilization of linear systems using coding and limited communication. *Proceedings of the 41st IEEE Conference on Decision and Control* 2002;

[16] Ling, Q, Lemmon MD. Stability of quantized control systems under dynamic bit assignment. *IEEE Transactions on Automatic Control* 2005; **50**(5): 734-740.

[17] Martins, NC, Dahleh MA. Feedback Control in the Presence of Noisy Channels: 'Bode-Like' Fundamental Limitations of Performance. *Scheduled to appear in the IEEE Transactions on Automatic Control in May 2008* Abridged version in ACC 2005.

[18] Martins, NC, Dahleh MA, Doyle JC, Fundamental Limitations of Disturbance Attenuation in the Presence of Side Information. *IEEE Transactions on automatic control* 2007; **52**(1).

[19] Nair, GN, Evans, RJ. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM Journal on Control and Optimization, Society for Industrial and Applied Mathematics, USA* 2004; **43**(2): 413 - 436.

[20] Proakis, JG. *Digital Communications* (4th edn). McGraw Hill, 2000.

[21] Sarma, SV, Dahleh, MA, Salapaka, S. On Time-Varying Bit-Allocation Maintaining Stability: A Convex Parameterization. *Proceedings of the 43rd IEEE Conference on Decision and Control* 2004.

[22] Sarma, S.V.; Dahleh, M.A. Real-Time Finite-Rate Tracking: Performance Limitations. *Proceedings of American Control Conference* June 2007.

[23] Tatikonda, S. Control Under Communication Constraints. *IEEE Transactions on Automatic Control* 2004; **49**(7): 1056 - 1068.

[24] Voulgaris, PG, Hadjicostis, CN, Touri, R. A Robust Control Approach to Perfect Reconstruction of Digital Signals. *IEEE Transactions on Signal Processing*: 2007; **55**(9): 4444-4457.