Input-to-State Stability of a Nonlinear Discrete-time System via $R-$cycles

Keith R. Santarelli  
Alexandre Megretski  
Munther A. Dahleh  
Laboratory for Information and Decision Systems  
Massachusetts Institute of Technology  
Cambridge, MA, 02139, USA  
Email: {krsanta, ameg, dahleh}@mit.edu

Abstract— The input-to-state stability of a particular nonlinear discrete-time system is investigated using a construct to which we refer as an $R-$cycle. Informally speaking, an $R-$cycle is a finite subsequence of a state trajectory for which the first and last elements of the sequence lie in a given set $R$. We first provide a formal definition of an $R-$cycle, along with appropriate sufficient conditions to guarantee the existence of $R-$cycles for the system under investigation. Next, we prove a useful bound on the Euclidean norm of the state for a single $R-$cycle. By then viewing the full state trajectory as a concatenation of $R-$cycles, we are then able to construct a bound on the $\ell_{\infty}$ norm of the full state trajectory.

I. INTRODUCTION

The problem of stabilizing a continuous-time plant via a hybrid feedback controller with a finite number of states is a problem that has received much attention in the recent literature (see, e.g., [1], [7], [8], and [9]). In this paper, we explore a method of assessing input-to-state stability for a particular nonlinear discrete-time system which originates from the analysis of a particular hybrid control scheme. The dynamics of the system can be described in the following manner: for each $k \in \mathbb{Z}^+$,

$$
\begin{align*}
    x[k+1] &= A(z_2[k])x[k] \\
    z_1[k+1] &= \text{sgn}(x_1[k] + w[k]) \\
    z_2[k+1] &= \begin{cases} 
        1 & z_1[k] = \text{sgn}(x_1[k] + w[k]), \\
        z_2[k] = 7000 & z_2[k] = 7000 \\
        1 + z_2[k] & \text{otherwise}
    \end{cases}
\end{align*}
$$

where $w[k] \in \mathbb{R}$ is an exogenous input, $x[k] = [x_1[k] \ x_2[k]]' \in \mathbb{R}^2$ and $z[k] = [z_1[k] \ z_2[k]]' \in \mathbb{Z}^+ \times \mathbb{Z}^+$ comprise the state, and

$$
    A(z_2) = \begin{cases} 
        \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T\right) & A_1, & z_2 \in [1, 6000] \\
        \exp\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T\right) & A_2, & z_2 \in [6001, 7000]
    \end{cases}
$$

where $T = \pi/8000$.

The structure and origin of the above description are closely related to the system whose block diagram is depicted in Fig. I.1. In this figure, a continuous-time LTI plant is placed in a feedback interconnection with a hybrid controller which operates in the following manner. Assuming for the moment that the exogenous inputs $v(t)$ and $w(t)$ are both identically 0 for all $t$, the output of the continuous-time plant $y(t)$ is sampled uniformly every $T$ seconds to produce a discrete-time signal $y_d[k]$. The signal $y_d[k]$ is then processed by a quantizer $Q(\cdot)$ to produce a discrete-time, discrete-amplitude signal $\tilde{y}_d[k]$ which is then input to the finite state automaton (FSA) with output $\tilde{g}_d[k]$. The zero-order hold (ZOH) creates a continuous-time signal $g(t)$ which is used to select the value of the gain at each time $t$ via the relation

$$
    g(t) = \tilde{g}_d[k], \quad kT \leq t < (k+1)T. \quad (I.5)
$$

The overall output of the hybrid controller is given by the relationship $u = K_g(t)y(t)$ where $K_{gi} \in \{K_1, K_2, \ldots, K_M\}$, $K_i \in \mathbb{R}, 1 \leq i \leq M$, where $M$ is the number of outputs of the FSA.

The block diagram of Fig. I.1 gives rise to the system described by Eqn. I.1, I.2, and I.3 when the continuous-time plant is a double integrator with state-space description

$$
    \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.
$$

For this plant, and for a particular choice of the quantizer $Q(\cdot)$ and FSA, we are interested in assessing the input-to-state stability of the system in Fig. I.1 when the input $v(t)$
is identically 0 and the input \( w(t) \) is bounded. It is clear, then, that Eqn. I.1 and I.4 describe the dynamics of the sampled plant state for a sampling interval of \( T = \pi/8000 \) for a scenario in which \( K_g \in \{-1, 1\} \). Eqn. I.2 and I.3 describe the dynamics of the FSA under consideration when \( Q(x) = \text{sgn}(x) \), and the discrete-time signal \( w[k] \) represents the bounded input.

The ultimate goal of this document is to prove the following assertion.

**Proposition I.1:** For any initial condition \( x[0] = [x_1[0], x_2[0]]' \), \( z[0] = [z_1[0], z_2[0]]' \) and any bounded input sequence \( w[k] \), the state \( x[k] \) described by the system of Eqn. I.1, I.2, I.3, and I.4 is bounded for all \( k \). Moreover, the \( \ell_\infty \) norm of the sequence \( x[k] \), denoted by \( ||x||_\infty \), has a bound of the form

\[
||x||_\infty \leq \max(\alpha ||x[0]||_2, \beta ||w||_\infty)
\]

where \( \alpha \) and \( \beta \) are positive real constants, and \( ||\cdot||_2 \) denotes Euclidean norm.

In addition to proving the above proposition, a method of determining explicit values for the constants \( \alpha \) and \( \beta \) will be constructed.

In contrast to the method we will use here, it is typical in problems such as this to attempt to find a storage function to prove stability. This, however, turns out to be an arduous task with little-to-no useful results. We introduce here a method of assessing input-to-state stability that circumvents the standard Lyapunov/storage techniques [5]. We begin by introducing a construct that we will refer to as an \( R-\text{cycle} \). Informally speaking, an \( R-\text{cycle} \) is a finite subsequence of a trajectory for which the beginning and end points lie within a certain region \( R \) of the state space, and for which at least one element in the subsequence does not lie in \( R \). The term \( R-\text{cycle} \) is, hence, motivated by the notion of the state leaving the set \( R \) and then returning back at a later time. By developing bounds on \( ||x[k]||_2 \) for \( R-\text{cycles} \) and by viewing system trajectories as concatenations of \( R-\text{cycles} \), we will be able to find \( \alpha \) and \( \beta \) which prove the existence of the bound of Proposition I.1.

It should be noted that, while much work has been performed in the recent literature on the study of input-to-state stability for discrete-time systems (see, for instance, [2], [3], [6]), all of this work has been formulated under the more traditional assumption that the system state \( x \) lies in \( \mathbb{R}^n \) and, thus, does not directly apply to the hybrid system described here in which the state has both a continuous component \( x \) and a discrete component \( z \). Recent work by Liberzon et. al. in [4] develops a technique very similar to the one we will use here, albeit for a different class of systems. In [4], the systems under consideration involve quantized state feedback, with a variable quantizer whose precision can be made arbitrarily accurate around 0; in the example we examine here, we develop a similar technique for quantized output feedback where the quantizer is fixed.

For convenience, we use the notation \( (x[k], z[k]) \) to refer to the concatenated state of the system. Also, without loss of generality, we prove the bound of Proposition I.1 for the case that \( ||w||_\infty = 1 \); the proof for an arbitrary bound is similar and is left to the reader.

We warn the reader at the forefront that many of the formal proofs presented here are rather technical in nature. Therefore, it is advised to ignore many of the proof details on a first pass in order to understand the basic ideas which lie behind the given technique.

II. CONSTRUCTION OF \( R-\text{CYCLES} \)

We begin with a definition.

**Definition II.1:** Consider a trajectory of the system described by Eqn. I.1, I.2, I.3, and I.4 and a set \( R \) defined as

\[
R = \{(x, z) : x \in V_1 \cup V_2, z_2 = 1\}, \quad (II.6)
\]

\[
V_1 = \{(x_1, x_2) : |x_1| \leq (1 + \epsilon)\}, \quad (II.7)
\]

\[
V_2 = \{(x_1, x_2) : x_2^2 \leq \gamma x_1^2\}. \quad (II.8)
\]

where \( \gamma > 0 \) and \( \epsilon = \epsilon(\gamma) > 0 \) are given constants. Suppose there exist two different times \( m_1 \) and \( m_2 \) with \( m_2 > m_1 \) such that the following conditions hold:

1. \( (x[m_i], z[m_i]) \in R \) for \( i = 1, 2 \).
2. There exists a value \( k \in (m_1, m_2) \) such that \( (x[k], z[k]) \notin R \).

Then the sequence \( C[i] = (x[m_i + i], z[m_i + i]) \) for \( i = 0, 1, \ldots, m_2 - m_1 \) is referred to as an \( R-\text{cycle} \) of the trajectory \( (x[k], z[k]) \).

A graphical depiction of the region \( \mathcal{V} = V_1 \cup V_2 \) is shown in Def. II.1 for some values of \( \gamma \) and \( \epsilon \) as depicted in Fig. II.2. In layman’s terms, the above definition defines an \( R-\text{cycle} \) as a portion of the state trajectory which lies in the region \( R \), leaves this region, and then returns back at some future time. The reasons for defining the set \( R \) (which is tailored to the specific example at hand) in the indicated manner may not be immediately obvious but will become more clear in later sections.

It is not immediately apparent that \( R-\text{cycles} \) exist, in general, for the trajectories of the system under investigation, a fact which we now show. We begin by establishing a useful property of the system trajectories.

![Graphical representation of the region \( \mathcal{V} = V_1 \cup V_2 \) of Def. II.1.](image-url)

**Proposition II.2:** Consider the system whose evolution is described by Eqn. I.1, I.2, I.3, and I.4. Then, for any
Proposition II.2 establishes the fact that all trajectories of this system must continually pass through a region \( \mathcal{R} \) when the parameters \( \gamma \) and \( \epsilon \) are chosen appropriately. While it is certainly possible that a trajectory can enter this region and never leave, if a trajectory does leave this region, it must return back at some future time, hence motivating the given definition of an \( \mathcal{R} \)-cycle. Moreover, the result of Proposition II.2 leads to a new way of viewing the system trajectories—as concatenations of cycles. We formalize this result in the following theorem.

Theorem II.1: Suppose that \( \gamma > 0 \) and \( \epsilon > 0 \) are chosen such that the conditions of Proposition II.2 are satisfied. Then for any trajectory of the system described by Eqn. I.1, I.2, I.3, and I.4 with arbitrary initial condition \((x[0], z[0])\) and input \(w[k]\) with \(\|w\|_\infty = 1\), exactly one of the following is true:

1) There exists a strictly increasing sequence of times \(\{j_i\}_{i=1}^\infty\) along with a set of sequences \(C_i[k]\) defined on the range \(0 \leq k \leq j_{i+1} - j_i\) for all \(i\) such that

\[
C_i[k - j_i] = (x[k], z[k]) \quad j_i \leq k \leq j_{i+1},
\]

\[
C_i[j_{i+1} - j_i] = C_{i+1}[0]
\]

for all \(i\) where each \(C_i[k]\) is an \(\mathcal{R}\)-cycle for the given \(\gamma\) and \(\epsilon\).

2) There exists a nonnegative integer \(N\) along with a sequence of times \(\{j_i\}_{i=1}^N\) and a (possibly empty) set of sequences \(C_i[k]\) defined on the range \(0 \leq k \leq j_{i+1} - j_i\) for \(i = 1, 2, \ldots, N\) such that

\[
C_i[k - j_i] = (x[k], z[k]) \quad j_i \leq k \leq j_{i+1},
\]

\[
C_i[j_{i+1} - j_i] = C_{i+1}[0]
\]

for all \(i = 1, 2, \ldots, N\) where each \(C_i[k]\) is an \(\mathcal{R}\)-cycle for the given \(\gamma\) and \(\epsilon\). Moreover, the trajectory \((x[k], z[k])\) is an \(\mathcal{R}\)-cycle for all \(k \geq j_{N+1}\).

Proof: Proposition II.2 guarantees for a given state trajectory an infinite sequence of times \(\{k_i\}_{i=1}^\infty\) for which the state trajectory must lie in \(\mathcal{R}\). Suppose that for every integer \(M\), there exists some \(k > M\) such that \(k\) is not contained in the sequence \(\{k_i\}_{i=1}^\infty\). Then it is clear that there exists a subsequence \(\{j_i\}_{i=1}^\infty\) of \(\{k_i\}_{i=1}^\infty\) such that

\[
(x[j_i], z[j_i]) \in \mathcal{R} \quad \text{for all } i
\]

For every \(i\), there exists \(k\) such that \(j_i < k < j_{i+1}\) and such that \((x[j_i], z[j_i]) \notin \mathcal{R}\).

In this case, the construction of the \(\mathcal{R}\)-cycles listed in item 1 immediately follows.

If, however, for a given trajectory, there exists an integer \(M\) for which every \(k \geq M\) is contained in the sequence \(\{k_i\}_{i=1}^\infty\), then \((x[k], z[k]) \in \mathcal{R} \quad \text{for all } k \geq M\). If the smallest integer \(M\) for which this holds is strictly greater than \(k_1\), then there exists a finite sequence of integers \(\{j_i\}_{i=1}^{N+1}\) of \(\{k_i\}_{i=1}^\infty\) for which

\[
(x[j_i], z[j_i]) \in \mathcal{R} \quad \text{for all } i = 1, 2, \ldots, N
\]

For every \(i = 1, 2, \ldots, N\), there exists \(k\) such that \(j_i < k < j_{i+1}\) and such that \((x[j_i], z[j_i]) \notin \mathcal{R}\).
and a construction of a finite set of $\mathcal{R}$-cycles listed in item 2 immediately follows. If, however, the smallest $M$ for which the given property holds is equal to $k_1$, then the state trajectory $(x[k], z[k]) \notin \mathcal{R}$ for all $k < M$, and $(x[k], z[k]) \in \mathcal{R}$ for all $k \geq M$, in which case no $\mathcal{R}$-cycles can be constructed.

In layman’s terms, Theorem II.1 essentially allows us to view any state trajectory as a concatenation of $\mathcal{R}$-cycles; in some cases, the trajectory may contain an infinite number of $\mathcal{R}$-cycles, while in others, it contains only a finite number.

By studying the behavior of the system trajectories over individual cycles, we can infer properties about the corresponding full trajectories. Specifically, by developing bounds on the Euclidean norm of $x[k]$ for a single $\mathcal{R}$-cycle, we will be able to develop the desired bound on $||x||_{\infty}$.

III. ANÁLISIS VIA $\mathcal{R}$-CYCLES

In order to develop bounds such as the one listed in Proposition I.1, we must first choose values of $\gamma$ and $\epsilon$ to construct a region $\mathcal{R}$ which possesses useful mathematical properties. In order to motivate the methodology which we use to determine $\gamma$ and $\epsilon$, we begin with some preliminary statements.

A. Preliminaries: Contractive Linear Transformations

We begin with the following proposition:

Proposition III.3: Consider the linear transformations $V_1 = A_1^{00} A_0^{00}$ and $V_2 = A_2^{00} A_0^{00}$. There exists $\gamma > 0$ such that $||V_1 x||_2 \leq ||x||_2$ and $||V_2 x||_2 \leq ||x||_2$ whenever $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ satisfies the conic relation $x_1^2 \leq \gamma x_2^2$.

Proof: To show that there exists a value of $\gamma > 0$ for which the above statement holds, first consider the case when $\gamma = 0$. In this case, the statement reduces to showing that $||V_1 x_0||_2 < ||x_0||_2$ and $||V_2 x_0||_2 < ||x_0||_2$, where $x_0$ takes the form $x_0 = \begin{bmatrix} 0 & \alpha \end{bmatrix}$. Simple calculations show that the both of these inequalities hold strictly for any value of $\alpha$.

To show that the above inequalities hold for some $\gamma > 0$, consider the case when $\gamma = 1$ and consider the vector $x_1 = \begin{bmatrix} \sin \phi & \cos \phi \end{bmatrix}$ for some $\phi \in (-\pi, \pi]$. Because $V_1$ and $V_2$ are continuous linear transformations, and because the Euclidean norm is a continuous function of its argument, it follows by continuity that, for $|\phi|$ sufficiently small, $||V_1 x_1||_2 \leq ||x_1||_2$ and $||V_2 x_1||_2 \leq ||x_1||_2$. Let $\phi_0$ be the maximum value of $\phi$ for which the the prior inequalities hold. If we utilize the fact that the transformations $V_1$ and $V_2$ are homogeneous, then it follows that $V_1$ and $V_2$ are contractions whenever $x_1^2 < \gamma x_2^2$ where $\gamma = \tan^2 \phi_0$.

The transformations $V_1$ and $V_2$ in Proposition III.3 can be interpreted in the following way: for a given vector $x$, $V_1 x$ is the vector which results 7000 time steps later when the state variable $z_2$ starts to increase, and $V_2 x$ is the vector that results 7001 time steps later if $z_2$ remains 1 for one additional time step and begins incrementing the time step thereafter. While use of these transformations may not immediately be apparent, the existence of $\gamma$ that satisfies such properties allows us to make a statement which will be useful:

Proposition III.4: Let $V_1$, $V_2$, and $\gamma$ be defined as in Proposition III.3. Consider the region $\mathcal{V}$ given by

$\mathcal{V} = V_1 \cup V_2$

$V_1 = \{(x_1, x_2) : |x_1| \leq A\}$

$V_2 = \{(x_1, x_2) : x_1^2 < \gamma x_2^2\}$

where $A > 0$. Then for each $x \in \mathcal{V}$,

$||V_1 x||_2 \leq \max\{||x||_2, B\}$

for $i = 1, 2$, where $B = A\sqrt{\gamma^{-1} + 1}$.

Proof: The proposition holds trivially for $x \in V_2$. Hence, the only points for which the proposition must be proved are those in the set difference $V_1 \backslash V_2$. Note that this region is compact. Moreover, because both $V_1$ and $V_2$ are homogeneous, the maximum of both $||V_1 x||$ and $||V_2 x||$ for $x$ in the indicated region occurs along the boundary $|x_1| = A, |x_2| \leq A/\sqrt{\gamma}$. It suffices to show that the maximum of each of these transformations along the boundary is less than or equal to $B$. To show this, note that $V_1$ can be written as

$V_i = \frac{1}{2} \begin{bmatrix} e^{T_1} + e^{-T_1} & e^{T_1} - e^{-T_1} \\ e^{-T_1} - e^{T_1} & e^{-T_1} + e^{T_1} \end{bmatrix} \begin{bmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{bmatrix}$

where $\theta_1 = 3\pi/4, \theta_2 = 6000\pi/8000, \text{ and } T_1 = 1000^T$. Applying this transformation to the vector $x_0 = [A \alpha ]^T$ where $\alpha \geq 0$, simple calculations show that $||V_i x_0||^2$ is increasing for $\alpha > 0$ so long as $\cos 2\theta_i \geq 0$. Since this is the case for both $\theta_1 = 3\pi/4$ and $\theta_2 = 6000\pi/8000$, we conclude that $||V_i x_0||^2$ is an increasing function of $\alpha > 0$ for both $i = 1, 2$, and, hence, the maximum values of $||V_i x_0||$ are $\alpha$ occurs where $\alpha = 1/\sqrt{\gamma}$. For, both of this value of $\alpha$, $||V_i x_0|| = ||x_0|| = B$. Using appropriate symmetry arguments, we can show that $B$ is an upper bound for $||V_i x||$ over the entire set $|x_1| \leq A, x_1^2 \geq \gamma x_2^2$, and, hence $||V_i x||_2 \leq B = \max\{||x||_2, B\}$ for $i = 1, 2$.

Proposition III.4 shows that a bound exists which is similar to the ultimate bound of Proposition I.1 that we wish to establish. Determining an actual value for the bound $B$ in Proposition III.4 (or, equivalently, $\gamma$ in Proposition III.3) can be performed using standard semi-definite programming techniques. Using a numerical toolbox such as MATLAB’s `icpcbeta` toolbox, we find that the largest value of $\gamma$ for which the constraints $||V_i x||_2 \leq ||x||_2, i = 1, 2$ can satisfied for $x_1^2 \leq \gamma x_2^2$ is $\gamma = 0.4547$. For the value $A = 1$, the corresponding value of $B$ is 1.7886.

B. Creating $\mathcal{R}$: Choosing $\gamma$ and $\epsilon$

Proposition III.4 will be used as our basis for analyzing $\mathcal{R}$-cycles. It should come as no surprise, then, that the value of $\gamma$ that we wish to use to create the region $\mathcal{R}$ is the value just calculated—the largest value such that
the linear maps $V_1$ and $V_2$ are contractions in the cone $x_1^T \leq \gamma x_2^T$, or, namely, $\gamma = 0.4547$. A simple calculation shows that this value of $\gamma$ satisfies the necessary lower bound imposed in the proof of Proposition II.2, hence, the results of Proposition II.2 and Thm. I.I hold. Calculations show that when $\gamma = 0.4547$, a choice of $\epsilon = 0.001$ will satisfy the necessary requirements for Proposition II.2 and Thm. I.I as well. If we define $\gamma^* = 0.4547$ and $\epsilon^* = 0.001$, then we can define a region $R^*$

$$R^* = \{(x, z) : x \in \mathcal{V}, z_2 = 1\}$$

$$\mathcal{V}^* = \{(x_1, x_2) : |x_1| \leq (1 + \epsilon^*)\} \cup \{(x_1, x_2) : x_2^2 \leq \gamma^* x_2^2\}.$$  

For convenience, we shall refer to $R^*$-cycles for the region $R^*$ as $R^*$-cycles, and we shall, henceforth, examine the boundedness properties of $R^*$-cycles.

**C. Boundedness Properties of $R^*$-cycles**

We begin by establishing the following result:

**Proposition III.5:** Consider an $R^*$-cycle $C[i]$, $i = 0, 1, \ldots, N$ of the system described by Eqn. I.1, I.2, I.3, and I.4 with input $w[k]$ that satisfies $||w||_\infty = 1$. Let $C[i] = (C[i], C[i])$, where $C[i]$ represents the portion of $\mathcal{C}[i]$ due to the state $x$, and $C[i]$ represents the portion of $\mathcal{C}[i]$ due to state $z$. Then the following holds:

$$||C[i][N]||_2 \leq \max\{||C[i][0]||_2, B\}$$  \hspace{1cm} (III.9)

where $B = (1 + \epsilon^*)\sqrt{(\gamma^*)^{-1} + 1}$.

**Proof:** To begin, note that any $R^*$-cycle of this system satisfies the constraint that

$$C[i][N] = A_1^M V_j A_1^M C[i][0]$$

for $j = 0, 1$ or 2 where $V_1$ and $V_2$ are defined as in Proposition III.3, and $V_0 = I_2$, the $2 \times 2$ identity matrix. Moreover, $M_1, M_2 \geq 0$, and $A_1^M C[i][0] \in R^*$. The case $j = 0$ corresponds to a trivial case in which the first element of $C[i]$ is constant for $i = 0, 1, \ldots, N$ and for which the second element of $C[i]$ (which represents the state variable $z_2$) is equal to 1 for $i = 0, 1, \ldots, N$. In this case we have $C[i][N] = A_1^M C[i][0]$.

The cases where $j = 1$ and $j = 2$ correspond to cases where $z_2$ (and hence the second component of $C[i]$) increments through the cycle. The term $A_1^M$ accounts for rotation of the state $x$ before $z_2$ begins to increment. Note in this situation that $z_2$ must begin incrementing either while the continuous portion of the state lies in the strip $|x_1| \leq 1$ or one time step after the continuous portion of the state leaves this strip. Hence, by choosing $V_1$ or $V_2$ as appropriate, we can always choose $M_i$ such that $A_1^M C[i][0] \in R^*$. Finally, the $A_1^M$ term represents possible additional rotation after $z_2$ has become 1 again in order to guarantee that $C[i][N] \in \mathcal{V}^*$ and, hence $C[i]\in R^*$.

For the case when $j = 0$, the statement follows trivially since $A_1$ is a rotation matrix and, hence, $||C[i][N]||_2 = ||A_1^N C[i][0]||_2 = ||C[i][0]||_2$. When $j = 1$ or $j = 2$, we have that

$$||C[i][N]||_2 = ||A_1^M V_j A_1^M C[i][0]||_2 = ||V_j A_1^M C[i][0]||_2.$$  

If we let $x = A_1^M C[i][0]$, then $x \in R^*$, and, hence,

$$||C[i][N]||_2 = ||V_j x||_2,$$  \hspace{1cm} $x \in R^*$

for which the given bound immediately follows as a result of Proposition III.4.

Proposition III.5 provides a useful bound relating the beginning of an $R^*$-cycle and the end of the same $R^*$-cycle. Since, however, we are interested in computing a bound on $||x||_\infty$ which involves providing a bound for all times (not just the beginning and end of $R^*$-cycles), a stronger bound is desired. The bound of Proposition III.5 can be extended by making use of the following basic Lemma, whose proof is immediate and is left to the reader.

**Lemma III.2:** Consider a linear transformation $A$ which, for a given vector $x$ satisfies $||Ax||_2 \geq ||x||_2$. Then $||A^2 x||_2 \geq ||Ax||_2$.

We now prove the following stronger assertion.

**Proposition III.6:** Consider an $R^*$-cycle $C[i], i = 0, 1, \ldots, N$ of the system described by Eqn. I.1, I.2, I.3, and I.4 with input $w[k]$ that satisfies $||w||_\infty = 1$. Let $C[i] = (C_1[i], C_2[i])$, where $C_1[i]$ represents the portion of $C[i]$ due to the state $x$, and $C_2[i]$ represents the portion of $C[i]$ due to the state $z$. Then the following holds:

$$\max_{i=0,1,\ldots,N} ||C[i][i]||_2 \leq \max\{||C[i][0]||_2, B\} \hspace{1cm} (III.10)$$

where $B = (1 + \epsilon^*)\sqrt{(\gamma^*)^{-1} + 1}$.

**Proof:** To begin, note that $C[i]$ takes on one of the following three forms for any $i = 0, 1, \ldots, N$:

$$C[i] = \begin{cases} C_1[i], & M_1 \geq 0 \\ C_2[i], & M_2 \geq 0 \end{cases}$$

$$C[i] = \begin{cases} A_1^M C[i][0], & M_1 \geq 6000, M_2 \leq 1000 \\ A_1^M A_2^{1000} C[i][0], & M_1 \geq 6000, M_2 \geq 0 \end{cases}$$

When $C[i]$ is of either the first or third listed forms, the result follows from the analysis of the proof in Proposition III.5. Hence, the desired bound need only be proved when $C[i]$ is of the second listed form.

We consider two cases: one in which $||C[i][N]||_2 \leq ||C[i][0]||_2$, and one in which $||C[i][N]||_2 > ||C[i][0]||_2 \geq ||C[i][0]||_2$. In the first case, let $x = A_1^M C[i][0]$, and note that $||C[i][0]||_2 = ||x||_2$. We wish to show that $||A_1^M x||_2 \leq ||x||_2$ for $M_2 \leq 1000$. Suppose that this assertion is not true, i.e. that there exists some $M_2 < 1000$ such that $||A_1^M x||_2 > ||x||_2$. Then repeated application of Lemma III.2 shows that

$$||A_1^{1000} x||_2 > ||x||_2.$$  

an obvious contradiction. Hence, it follows that if $||C[i][N]||_2 \leq ||C[i][0]||_2$ then $||C[i][i]||_2 \leq ||C[i][0]||_2$.

For all $i = 0, 1, \ldots, N$.  \hspace{1cm} WeA10.1
In the second case, again let \( x = A^M_2 C_2[0] \), and note that \( |C_2[N]|_2 = |A^0_{2,000} x|_2 > |x|_2 \). We show now that \( |A^M_{2,2} x|_2 \leq |A^0_{2,000} x|_2 \) for all \( 0 \leq M_2 \leq 1000 \). Assume that there exists \( 0 \leq M_2 < 1000 \) such that \( |A^M_{2,2} x|_2 > |A^0_{2,000} x|_2 \). Then \( |A^M_{2,2} x|_2 > |x|_2 \), and by Lemma III.2, \( |A^0_{2,000} x|_2 \geq |A^M_{2,2} x|_2 \). From this result, it then follows that if \( |C_2[N]|_2 \leq B \), then \( |C_2[i]|_2 \leq B \) for all \( i = 0, 1, \ldots, N \). Combining both the first and second cases, we find that \( |C_2[i]|_2 \leq \max\{|C_2[0]|_2, B\} \) for all \( i = 0, 1, \ldots, N \) as desired.

\[\text{IV. Main Result}\]

Proposition III.6 provides a useful bound on the behavior of the continuous portion of individual cycles. Since, by Thm. II.1, we can view full state trajectories as concatenations of cycles, we can use the boundedness result for individual cycles to now prove the ultimate bound on \( |x|_\infty \). The remainder of this section is devoted to that proof.

Proof: [Proof of Proposition I.1] Recall from Thm. II.1 that there exists a finite or infinite sequence of times \( \{j_k\} \) which mark the beginnings and ends of cycles. We will first show the following intermediate bound: for each trajectory \((x[k], z[k])\) of the system with input \( w[k] \) satisfying \( ||w||_\infty = 1 \),

\[||x[k]||_2 \leq \max\{|||x[j_1][j_2], B\}, k \geq j_1, \]

Proposition III.6 proves this assertion for \( j_1 \leq k \leq j_2 \). Assume that the statement holds for all times up to \( j_k \), i.e.

\[||x[k]||_2 \leq \max\{|||x[j_1][j_2], B\}, j_1 \leq k \leq j_k.\]

Now, for \( j_k \leq k \leq j_{k+1} \),

\[||x[k]||_2 \leq \max\{|||x[j_1][j_2], B\}, j_1 \leq k \leq j_{k+1}.\]

If the sequence \( \{j_k\} \) is infinite, then the given bound holds for all \( k \geq j_1 \). If the sequence is finite, then there exists some time \( j_N \) such that \((x[k], z[k]) \in \mathcal{R}^\ast \) for all \( k \geq j_N \). This, however, implies that \( |x[k]|_2 \leq (1 + \epsilon^\ast) \) for all \( k \geq j_N \) (since \( z_2[k] = 1 \) for all \( k \geq j_N \), \( |x[k]|_2 \) is constant for \( k = j_N \), and if \( |x[j_N]|_2 > (1 + \epsilon^\ast) \), \( x[k] \notin \mathcal{V}^\ast \) for some \( k \geq j_N \)). Since \( (1 + \epsilon^\ast) B \), the bound for \( k \geq j_1 \) follows.

What remains is to find a constant \( \alpha \) such that \( |x[j_1]|_2 \leq \alpha \|x[0]\|_2 \) for any trajectory of the system with \( ||w||_\infty = 1 \). This can be performed by engaging an analysis similar to that of the proof of Proposition II.2. Without loss of generality, consider the case where the initial condition \((x[0], z[0]) \notin \mathcal{R}^\ast \). We shall first consider the case where \( z[0] = 0 \). In this case, there exists some \( 0 < k_1 < 7000 \) such that \( z_2[k_1] = 7000 \) and \( x[k_1] = A^0_{1000} A^M_1 0 \) for some \( M_1 \geq 0 \). It follows that

\[|x[k_1]|_2 \leq |A^0_{2,000} x[0]|_2 \leq \exp\left(\frac{\pi}{8}\right) |x[0]|_2.\]

If \( x[k_1] \in \mathcal{V}^\ast \), then we may take \( j_1 = k_1 \). Otherwise, \( z_2[k] \) will remain 1 for \( k_1 \leq k \leq k_2 \) where \( x[k_2] \) satisfies one of the following two conditions:

1) \(|x[k_2]|_2 \leq 1 + \epsilon^\ast \)
2) \(x[k_2] \leq \gamma^\ast x[k_2] \)

In either case, \( x[k_2] \in \mathcal{V}^\ast \) and, hence, we may take \( j_1 = k_2 \) with the bound

\[||x[j_1]|_2 \leq \exp\left(\frac{\pi}{8}\right) |x[0]|_2.\]

If \( z_2[0] = 1 \) and \( x[0] \notin \mathcal{V}^\ast \), an analysis similar to above shows that the above bound holds in this case as well so that, overall, for any trajectory of the system with \( ||w||_\infty = 1 \),

\[||x[j_1]|_2 \leq \exp\left(\frac{\pi}{8}\right) |x[0]|_2.\]

Now, using the fact that \( |x[k]|_2 \geq |x[k]|_\infty \), we find

\[|x|_\infty \leq \max\{\exp\left(\frac{\pi}{8}\right) |x[0]|_2, (1 + \epsilon^\ast) \sqrt{(\gamma^\ast)^{-1} + 1}\}.\]

\[\text{V. Concluding Remarks}\]

While the \( \mathcal{R} \)-cycles constructed here applied to a very specific case study, preliminary work indicates that for a large class of systems with the structure depicted in Fig. I.1, \( \mathcal{R} \)-cycles can be used to establish input-to-state stability when either or both inputs \( v(t) \) and \( u(t) \) are present, and can also be used to establish asymptotic stability of the sampled continuous state \( x[k] \) when both \( v(t) \) and \( u(t) \) are identically 0. Note that while the formal proofs involving \( \mathcal{R} \)-cycles to prove stability may be somewhat lengthy, the computational portion of finding the parameters \( \alpha \) and \( \beta \) in the upper bound are not taxing since it only requires solving a simple semi-definite program. Hence, for cases when they are applicable, \( \mathcal{R} \)-cycles can provide a computationally efficient way of computing a stability measure for a system.

\[\text{References}\]


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