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Input-to-State Stability of a Nonlinear Discrete-time System via $\mathcal{R}-$cycles

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***Abstract*— The input-to-state stability of a particular nonlinear discrete-time system is investigated using a construct to which we refer as an $\mathcal{R}-$cycle. Informally speaking, an $\mathcal{R}-$cycle is a finite subsequence of a state trajectory for which the first and last elements of the subsequence lie in a given set $\mathcal{R}$. We first provide a formal definition of an $\mathcal{R}-$cycle, along with appropriate sufficient conditions to guarantee the existence of $\mathcal{R}-$cycles for the system under investigation. Next, we prove a useful bound on the Euclidean norm of the state for a single $\mathcal{R}-$cycle. By then viewing the full state trajectory as a concatenatation of $\mathcal{R}-$cycles, we are then able to construct a bound on the ℓ_∞ norm of the full state trajectory.**

I. Introduction

The problem of stabilizing a continuous-time plant via a hybrid feedback controller with a finite number of states is a problem that has received much attention in the recent literature (see, e.g., [1], [7], [8], and [9]). In this paper, we explore a method of assessing input-to-state stability for a particular nonlinear discrete-time system which originates from the analysis of a particular hybrid control scheme. The dynamics of the system can be described in the following manner: for each $k \in \mathbf{Z}^+$,

$$x[k+1] = A(z_2[k])x[k] \tag{I.1}$$

$$z_1[k+1] = \operatorname{sgn}(x_1[k]+w[k]) \tag{I.2}$$

$$z_2[k+1] = \begin{cases} 1 & z_1[k] = \operatorname{sgn}(x_1[k]+w[k]), \\ & \text{and } \ z_2[k]=1 \\ 1 & z_2[k] = 7000 \\ 1+z_2[k] & \text{otherwise} \end{cases} \tag{I.3}$$

where $w[k] \in \mathbf{R}$ is an exogenous input, $x[k] = \begin{bmatrix} x_1[k] & x_2[k] \end{bmatrix}' \in \mathbf{R}^2$ and $z[k] = \begin{bmatrix} z_1[k] & z_2[k] \end{bmatrix}' \in \mathbf{Z}^+ \times \mathbf{Z}^+$ comprise the state, and

$$A(z_2) = \begin{cases} \exp\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T\right) \triangleq A_1, & z_2 \in [1, 6000] \\ \exp\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T\right) \triangleq A_2, & z_2 \in [6001, 7000] \end{cases} \tag{I.4}$$

where $T = \pi/8000$.

The structure and origin of the above description are closely related to the system whose block diagram is depicted in Fig. I.1. In this figure, a continuous-time LTI plant is placed in a feedback interconnection with a hybrid

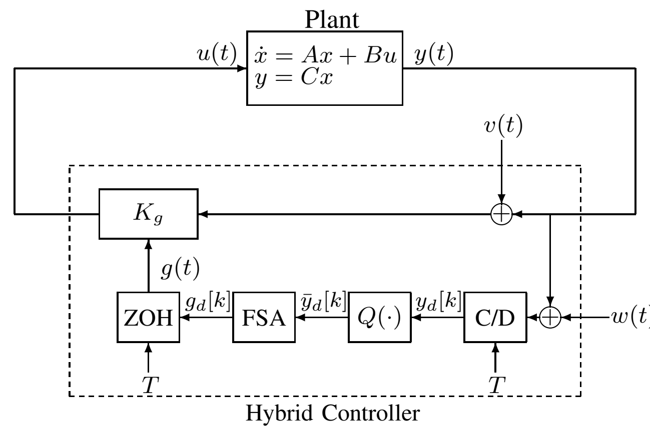

Fig. I.1. Hybrid system which motivates the discrete-time system described by Eqn. I.1, I.2, and I.3.

controller which operates in the following manner. Assuming for the moment that the exogenous inputs $v(t)$ and $w(t)$ are both identically 0 for all t, the output of the continuous-time plant $y(t)$ is sampled uniformly every T seconds to produce a discrete-time signal $y_d[k]$. The signal $y_d[k]$ is then processed by a quantizer $Q(\cdot)$ to produce a discrete-time, discrete-amplitude signal $\bar{y}_d[k]$ which is then input to a finite state automaton (FSA) with output $g_d[k]$. The zero-order hold (ZOH) creates a continuous-time signal $g(t)$ which is used to select the value of the gain at every time t via the relation

$$g(t) = g_d[k], \qquad kT \le t < (k+1)T. \tag{I.5}$$

The overall output of the hybrid controller is given by the relationship $u = K_{g(t)}y(t)$ where $K_g \in \{K_1, K_2, \ldots, K_M\}$, $K_i \in \mathbf{R}, 1 \le i \le M$, where M is the number of outputs of the FSA.

The block diagram of Fig. I.1 gives rise to the system described by Eqn. I.1, I.2, and I.3 when the continuous-time plant is a double integrator with state-space description

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

For this plant, and for a particular choice of the quantizer $Q(\cdot)$ and FSA, we are interested in assessing the input-to-state stability of the system in Fig. I.1 when the input $v(t)$

is identically 0 and the input $w(t)$ is bounded. It is clear, then, that Eqn. I.1 and I.4 describe the dynamics of the sampled plant state for a sampling interval of $T = \pi/8000$ for a scenario in which $K_g \in \{-1, 1\}$. Eqn. I.2 and I.3 describe the dynamics of the FSA under consideration when $Q(x) = \mathrm{sgn}(x)$, and the discrete-time signal $w[k]$ represents the bounded input.

The ultimate goal of this document is to prove the following assertion.

Proposition I.1: For any initial condition $x[0] = \begin{bmatrix} x_1[0] & x_2[0] \end{bmatrix}'$, $z[0] = \begin{bmatrix} z_1[0] & z_2[0] \end{bmatrix}'$ and any bounded input sequence $w[k]$, the state $x[k]$ described by the system of Eqn. I.1, I.2, I.3, and I.4 is bounded for all k. Moreover, the ℓ_∞ norm of the sequence $x[k]$, denoted by $||x||_\infty$, has a bound of the form

$$||x||_\infty \leq \max\{\alpha ||x[0]||_2,\ \beta ||w||_\infty\}$$

where α and β are positive real constants, and $||\cdot||_2$ denotes Euclidean norm.

In addition to proving the above proposition, a method of determining explicit values for the constants α and β will be constructed.

In contrast to the method we will use here, it is typical in problems such as this to attempt to find a storage function to prove stability. This, however, turns out to be an arduous task with little-to-no useful results. We introduce here a method of assessing input-to-state stability that circumvents the standard Lyapunov/storage techniques [5]. We begin by introducing a construct that we will refer to as an $\mathcal{R}-$*cycle*. Informally speaking, an $\mathcal{R}-$cycle is a finite subsequence of a trajectory for which the beginning and end points lie within a certain region $\mathcal{R}$ of the state space, and for which at least one element in the subsequence does *not* lie in $\mathcal{R}$. The term $\mathcal{R}-$cycle is, hence, motivated by the notion of the state leaving the set $\mathcal{R}$ and then returning back at a later time. By developing bounds on $||x[k]||_2$ for $\mathcal{R}-$cycles and by viewing system trajectories as concatenatations of $\mathcal{R}-$cycles, we will be able to find α and β which prove the existence of the bound of Proposition I.1.

It should be noted that, while much work has been performed in the recent literature on the study of input-to-state stability for discrete-time systems (see, for instance, [2], [3], [6]), all of this work has been formulated under the more traditional assumption that the system state x lies in $\mathbf{R}^n$ and, thus, does not directly apply to the hybrid system described here in which the state has both a continuous component x and a discrete component z. Recent work by Liberzon et. al. in [4] develops a technique very similar to the one we will use here, albeit for a different class of systems. In [4], the systems under consideration involve quantized *state* feedback, with a variable quantizer whose precision can be made arbitrarily accurate around 0; in the example we examine here, we develop a similar technique for quantized *output* feedback where the quantizer is fixed.

For convenience, we use the notation $(x[k], z[k])$ to refer to the concatenated state of the system. Also, without loss of generality, we prove the bound of Proposition I.1 for the case that $||w||_\infty = 1$; the proof for an arbitrary bound is similar and is left to the reader.

We warn the reader at the forefront that many of the formal proofs presented here are rather technical in nature. Therefore, it is advised to ignore many of the proof details on a first pass in order to understand the basic ideas which lie behind the given technique.

II. Construction of $\mathcal{R}-$cycles

We begin with a definition.

Definition II.1: Consider a trajectory of the system described by Eqn. I.1, I.2, I.3, and I.4 and a set $\mathcal{R}$ defined as

$$\mathcal{R} = \{(x, z) : x \in \mathcal{V}_1 \cup \mathcal{V}_2, z_2 = 1\}, \tag{II.6}$$

$$\mathcal{V}_1 = \{(x_1, x_2) : |x_1| \leq (1 + \epsilon)\} \tag{II.7}$$

$$\mathcal{V}_2 = \{(x_1, x_2) : x_1^2 \leq \gamma x_2^2\} \tag{II.8}$$

where $\gamma > 0$ and $\epsilon = \epsilon(\gamma) > 0$ are given constants. Suppose there exist two different times m_1 and m_2 with $m_2 > m_1$ such that the following conditions hold:

1) $(x[m_i], z[m_i]) \in \mathcal{R}$ for $i = 1, 2$.
2) There exists a value $k \in (m_1, m_2)$ such that $(x[k], z[k]) \notin \mathcal{R}$.

Then the sequence $\mathcal{C}[i] = (x[m_1 + i], z[m_1 + i])$ for $i = 0, 1, \ldots, m_2 - m_1$ is referred to as an $\mathcal{R}-$*cycle* of the trajectory $(x[k], z[k])$.

A graphical depiction of the region $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ described in Def. II.1 for some values of γ and ϵ is depicted in Fig. II.2. In layman's terms, the above definition defines an $\mathcal{R}-$cycle as a portion of the state trajectory which lies in the region $\mathcal{R}$, leaves this region, and then returns back at some future time. The reasons for defining the set $\mathcal{R}$ (which is tailored to the specific example at hand) in the indicated manner may not be immediately obvious but will become more clear in later sections.

It is not immediately apparent that $\mathcal{R}-$cycles exist, in general, for the trajectories of the system under investigation, a fact which we now show. We begin by establishing a useful property of the system trajectories.

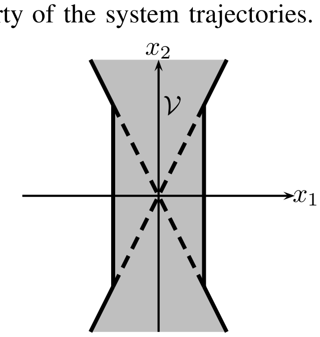

Fig. II.2. Graphical representation of the region $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ of Def. II.1.

Proposition II.2: Consider the system whose evolution is described by Eqn. I.1, I.2, I.3, and I.4. Then, for any sequence $w[k]$ with $||w||_\infty = 1$, and for any initial condition $(x[0], z[0])$, there exists a strictly increasing sequence $\{k_i\}_{i=1}^\infty$ and a region $\mathcal{R}$ (as described in Def. II.1) for which $(x[k_i], z[k_i]) \in \mathcal{R}$ for all i whenever $\gamma > 0$ and $\epsilon > 0$ of Eqn. II.7 and II.8 are chosen sufficiently large.

Before proving this statement, we need the result of the following lemma whose proof is left to the reader:

Lemma II.1: Suppose that a trajectory of the system described by Eqn. I.1, I.2, I.3, and I.4 satisfies the condition that, for some $m > 0$, $z_2[m-1] = 7000$ (and, hence, $z_2[m] = 1$). Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ of Eqn. II.7 and II.8. Then there exists $\epsilon > 0$ such that if $x[m] \notin \mathcal{V}$, then

$$\mathrm{sgn}(x_1[m-1] + w[m-1]) = \mathrm{sgn}(x_1[m] + w[m])$$

for all choices of $|w[m]| \le 1$ and $|w[m-1]| \le 1$.

Proof: [Proof of Proposition II.2] We will begin by choosing γ and ϵ to construct the set $\mathcal{V}$, and, hence, the set $\mathcal{R}$. First, we will choose γ such that $\gamma > \max\{\tanh^2 T, \tan^2 T\}$. By construction, there exists a choice of ϵ such that the result of Lemma II.1 holds, and we choose ϵ accordingly.

We now show existence of k_1 which satisfies $(x[k_1], z[k_1]) \in \mathcal{R}$. If $x[0] \in \mathcal{V}$ and $z_2[0] = 1$, we may take $k_1 = 0$. Otherwise, assume that at least one of these two conditions does not hold. If $z_2[0] \neq 1$, then $z_2[m_1] = 1$ for some $m_1 < 7000$. If $x[m_1] \in \mathcal{V}$, then we may take $k_1 = m_1$.

If $x[m_1] \notin \mathcal{V}$, then by the result of Lemma II.1, we have that $\mathrm{sgn}(x_1[m_1]+w[m_1]) = \mathrm{sgn}(x_1[m_1-1]+w[m_1-1])$ for all $|w[m_1]| \le 1$ and $|w[m_1-1]| \le 1$. This, in turn, implies that $z_2[m_1+1] = 1$. Because of the rotational nature of the matrix A_1, $z_2[k]$ will remain 1 for $m_1 \le k \le m_2$, where $m_2 < 15000$ satisfies at least one of two conditions:

1) $|x_1[m_2]| \le 1 + \epsilon$.
2) $x_1^2[m_2] \le \gamma x_2^2[m_2]$.

Note that if 1 does not occur, 2 must occur via construction of γ. In either case, we have that $x[m_2] \in \mathcal{V}$ and, hence, we may take $k_1 = m_2$.

If $z_2[0] = 1$ and $x[0] \notin \mathcal{V}$, then one of two possible scenarios can occur:

1) $z_2[k] = 1$ for $0 \le k \le m_1$ for some $0 < m_1 < 8000$, such that $x[m_1] \in \mathcal{V}$.
2) Item 1 does not occur, then $z_2[k] > k + 1$ for $k \le 6999$. Hence, $z_2[7000] = 1$. If $x[7000] \in \mathcal{V}$, take $k_1 = 7000$. Otherwise, we can find $k_1 < 15000$ such that $z_2[k_1] = 1, x[k_1] \in \mathcal{V}$ for $k_1 < 15000$ via Lemma II.1 using an argument similar to the one presented above.

Now that we have established the existence of k_1 with the desired properties, the time-invariant nature of this system implies that the same arguments can be made to show existence of $k_2 > k_1$ by applying the same argument for new initial conditions $\tilde{x}[0]$ and $\tilde{z}[0]$ given by $\tilde{x}[0] = x[k_1+1]$, $\tilde{z}[0] = z[k_1+1]$. Hence, via induction, the sequence $\{k_i\}_{i=1}^\infty$ exists with the desired properties. ■

Proposition II.2 establishes the fact that all trajectories of this system must continually pass through a region $\mathcal{R}$ when the parameters γ and ϵ are chosen appropriately. While it is certainly possible that a trajectory can enter this region and never leave, if a trajectory does leave this region, it *must* return back at some future time, hence motiviating the given definition of an $\mathcal{R}-$cycle. Moreover, the result of Proposition II.2 leads to a new way of viewing the system trajectories—as concatenations of cycles. We formalize this result in the following theorem.

Theorem II.1: Suppose that $\gamma > 0$ and $\epsilon > 0$ are chosen such that the conditions of Proposition II.2 are satisfied. Then for any trajectory of the system described by Eqn. I.1, I.2, I.3, and I.4 with arbitrary initial condition $(x[0], z[0])$ and input $w[k]$ with $||w||_\infty = 1$, exactly one of the following is true:

1) There exists a strictly increasing sequence of times $\{j_i\}_{i=1}^\infty$ along with a set of sequences $\mathcal{C}_i[k]$ defined on the range $0 \le k \le j_{i+1} - j_i$ for all i such that

$$\begin{aligned} \mathcal{C}_i[k - j_i] &= (x[k], z[k]) \qquad j_i \le k \le j_{i+1}, \\ \mathcal{C}_i[j_{i+1} - j_i] &= \mathcal{C}_{i+1}[0] \end{aligned}$$

for all i where each $\mathcal{C}_i[k]$ is an $\mathcal{R}-$cycle for the given γ and ϵ.

2) There exists a nonnegative integer N along with a sequence of times $\{j_i\}_{i=1}^{N+1}$ and a (possibly empty) set of sequences $\mathcal{C}_i[k]$ defined on the range $0 \le k \le j_{i+1} - j_i$ for $i = 1, 2, \ldots, N$ such that

$$\begin{aligned} \mathcal{C}_i[k - j_i] &= (x[k], z[k]) \qquad j_i \le k \le j_{i+1}, \\ \mathcal{C}_i[j_{i+1} - j_i] &= \mathcal{C}_{i+1}[0] \end{aligned}$$

for all $i = 1, 2, \ldots, N$ where each $\mathcal{C}_i[k]$ is an $\mathcal{R}-$cycle for the given γ and ϵ. Moreover, the trajectory $(x[k], z[k]) \in \mathcal{R}$ for all $k \ge j_{N+1}$.

Proof: Proposition II.2 guarantees for a given state trajectory an infinite sequence of times $\{k_i\}_{i=1}^\infty$ for which the state trajectory must lie in $\mathcal{R}$. Suppose that for every integer M, there exists some $k > M$ such that k is *not* contained in the sequence $\{k_i\}_{i=1}^\infty$. Then it is clear that there exists a subsequence $\{j_i\}_{i=1}^\infty$ of $\{k_i\}_{i=1}^\infty$ such that

- $(x[j_i], z[j_i])' \in \mathcal{R}$ for all i
- For every i, there exists k such that $j_i < k < j_{i+1}$ and such that $(x[j_i], z[j_i]) \notin \mathcal{R}$.

In this case, the construction of the $\mathcal{R}-$cycles listed in item 1 immediately follows.

If, however, for a given trajectory, there exists an integer M for which *every* $k \ge M$ is contained in the sequence $\{k_i\}_{i=1}^\infty$, then $(x[k], z[k]) \in \mathcal{R}$ for all $k \ge M$. If the smallest integer M for which this holds is strictly greater than k_1, then there exists a finite sequence of integers $\{j_i\}_{i=1}^{N+1}$ of $\{k_i\}_{i=1}^\infty$ for which

- $(x[j_i], z[j_i]) \in \mathcal{R}$ for all $i = 1, 2, \ldots, N$
- For every $i = 1, 2, \ldots, N$, there exists a k such that $j_i < k < j_{i+1}$ and such that $(x[j_i], z[j_i]) \notin \mathcal{R}$

and a construction of a finite set of $\mathcal{R}-$cycles listed in item 2 immediately follows. If, however, the smallest M for which the given property holds is equal to k_1, then the state trajectory $(x[k], z[k]) \notin \mathcal{R}$ for all $k < M$, and $(x[k], z[k]) \in \mathcal{R}$ for all $k \geq M$, in which case no $\mathcal{R}-$cycles can be constructed. ■

In layman's terms, Theorem II.1 essentially allows us to view any state trajectory as a concatenation of $\mathcal{R}-$cycles; in some cases, the trajectory may contain an infinite number of $\mathcal{R}-$cycles, while in others, it contains only a finite number.

By studying the behavior of the system trajectories over individual cycles, we can infer properties about the corresponding full trajectories. Specifically, by developing bounds on the Euclidean norm of $x[k]$ for a single $\mathcal{R}-$cycle, we will be able to develop the desired bound on $||x||_\infty$.

III. Analysis via $\mathcal{R}-$cycles

In order to develop bounds such as the one listed in Proposition I.1, we must first choose values of γ and ϵ to construct a region $\mathcal{R}$ which possesses useful mathematical properties. In order to motivate the methodology which we use to determine γ and ϵ, we begin with some preliminary statements.

A. Preliminaries: Contractive Linear Transformations

We begin with the following proposition:

Proposition III.3: Consider the linear transformations $V_1 = A_2^{1000} A_1^{6000}$ and $V_2 = A_2^{1000} A_1^{6001}$. There exists $\gamma > 0$ such that $||V_1 x||_2 \leq ||x||_2$ and $||V_2 x||_2 \leq ||x||_2$ whenever $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}'$ satisfies the conic relation $x_1^2 \leq \gamma x_2^2$.

Proof: To show that existence of a value of $\gamma > 0$ for which the above statement holds, first consider the case when $\gamma = 0$. In this case, the statement reduces to showing that $||V_1 x_0||_2 < ||x_0||_2$ and $||V_2 x_0||_2 < ||x_0||_2$, where x_0 takes the form $x_0 = \begin{bmatrix} 0 & \alpha \end{bmatrix}'$. Simple calculations show that the both of these inequalities hold strictly for any value of α.

To show that the above inequalities hold for some $\gamma > 0$, consider the case when $\alpha = 1$ and consider the vector $x_1 = \begin{bmatrix} \sin\phi & \cos\phi \end{bmatrix}'$ for some $\phi \in (-\pi, \pi]$. Because V_1 and V_2 are continuous linear transformations, and because the Euclidean norm is a continuous function of its argument, it follows by continuity that, for $|\phi|$ sufficiently small, $||V_1 x_1||_2 \leq ||x_1||_2$ and $||V_2 x_1||_2 \leq ||x_1||_2$. Let ϕ_0 be the maximum value of ϕ for which the the prior inequalities hold. If we utilize the fact that the transformations V_1 and V_2 are homogeneous, then it follows that V_1 and V_2 are contractions whenever $x_1^2 < \gamma x_2^2$ where $\gamma = \tan^2 \phi_0$. ■

The transformations V_1 and V_2 in Proposition III.3 can be interpreted in the following way: for a given vector x, $V_1 x$ is the vector which results 7000 time steps later when the state variable z_2 starts to increment, and $V_2 x$ is the vector that results 7001 time steps later if z_2 remains 1 for one additional time step and begins incrementing the time step thereafter. While use of these transformations may not immediately be apparent, the existence of γ that satisfies such properties allows us to make a statement which will be useful:

Proposition III.4: Let V_1, V_2, and γ be defined as in Proposition III.3. Consider the region $\mathcal{V}$ given by

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_1 \cup \mathcal{V}_2 \\ \mathcal{V}_1 &= \{(x_1, x_2) : |x_1| \leq A\} \\ \mathcal{V}_2 &= \{(x_1, x_2) : x_1^2 < \gamma x_2^2\} \end{aligned}$$

where $A > 0$. Then for each $x \in \mathcal{V}$,

$$||V_i x||_2 \leq \max\{||x||_2, B\}$$

for $i = 1, 2$, where $B = A\sqrt{\gamma^{-1} + 1}$.

Proof: The proposition holds trivially for $x \in \mathcal{V}_2$. Hence, the only points for which the proposition must be proved are those in the set difference $\mathcal{V}_1 \backslash \mathcal{V}_2$. Note that this region is compact. Moreover, because both V_1 and V_2 are homogeneous, the maximum of both $||V_1 x||$ and $||V_2 x||$ for x in the indicated region occurs along the boundary $|x_1| = A, |x_2| \leq A/\sqrt{\gamma}$. It suffices to show that the maximum of each of these transformations along the boundary is less than or equal to B. To show this, note that V_i can be written as

$$V_i = \frac{1}{2} \begin{bmatrix} e^{T_1} + e^{-T_1} & e^{T_1} - e^{-T_1} \\ e^{T_1} - e^{-T_1} & e^{T_1} + e^{-T_1} \end{bmatrix} \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix},$$

where $\theta_1 = 3\pi/4$, $\theta_2 = 6001\pi/8000$, and $T_1 = 1000T$. Applying this transformation to the vector $x_0 = \begin{bmatrix} A & A\alpha \end{bmatrix}'$ where $\alpha \geq 0$, simple calculations show that $||V_i x_0||^2$ is increasing for $\alpha > 0$ so long as $\cos 2\theta_i \geq 0$. Since this is the case for both $\theta_1 = 3\pi/4$ and $\theta_2 = 6001\pi/8000$, we conclude that $||V_i x_0||^2$ is an increasing function of $\alpha > 0$ for both $i = 1, 2$, and, hence, the maximum values of $||V_i x_0||_2$ over the constrained set $\alpha \leq 1/\sqrt{\gamma}$ (or, equivalently, $|x_2| \leq A\alpha$) occurs where $\alpha = 1/\sqrt{\gamma}$. But, for this value of α, $||V_i x_0|| \leq ||x_0|| = B$. Using appropriate symmetry arguments, we can show that B is an upper bound for $||V_i x||$ over the entire set $|x_1| \leq A$, $x_1^2 \geq \gamma x_2^2$, and, hence $||V_i x||_2 \leq B = \max\{||x||_2, B\}$ for $i = 1, 2$. ■

Proposition III.4 shows that a bound exists which is similar to the ultimate bound of Proposition I.1 that we wish to establish. Determining an actual *value* for the bound B in Proposition III.4 (or, equivalently, γ in Proposition III.3) can be performed using standard semi-definite programming techniques. Using a numerical toolbox such as MATLAB's `iqc_beta` toolbox, we find that the largest value of γ for which the constraints $||V_i x||_2 < ||x||_2, i = 1, 2$ can satisfied for $x_1^2 \leq \gamma x_2^2$ is $\gamma = 0.4547$. For the value $A = 1$, the corresponding value of B is 1.7886.

B. Creating $\mathcal{R}$: Choosing γ and ϵ

Proposition III.4 will be used as our basis for analyzing $\mathcal{R}-$cycles. It should come as no surprise, then, that the value of γ that we wish to use to create the region $\mathcal{R}$ is the value just calculated—the largest value such that the linear maps V_1 and V_2 are contractions in the cone $x_1^2 \leq \gamma x_2^2$, or, namely, $\gamma = 0.4547$. A simple calculation shows that this value of γ satisfies the necessary lower bound imposed in the proof of Proposition II.2, hence, the results of Proposition II.2 and Thm. II.1 hold. Calculations show that when $\gamma = 0.4547$, a choice of $\epsilon = 0.001$ will satisfy the necessary requirements for Proposition II.2 and Thm. II.1 as well. If we define $\gamma^* = 0.4547$ and $\epsilon^* = 0.001$, then we can define a region $\mathcal{R}^*$

$$\begin{aligned} \mathcal{R}^* &= \{(x, z) : x \in \mathcal{V}, z_2 = 1\} \\ \mathcal{V}^* &= \{(x_1, x_2) : |x_1| \leq (1 + \epsilon^*)\} \cup \\ & \quad \{(x_1, x_2) : x_1^2 \leq \gamma^* x_2^2\}. \end{aligned}$$

For convenience, we shall refer to $\mathcal{R}-$cycles for the region $\mathcal{R}^*$ as $\mathcal{R}^*-$*cycles*, and we shall, henceforth, examine the boundedness properties of $\mathcal{R}^*-$cycles.

C. Boundedness Properties of $\mathcal{R}^*-$cycles

We begin by establishing the following result:

Proposition III.5: Consider an $\mathcal{R}^*-$cycle $\mathcal{C}[i]$, $i = 0, 1, \ldots N$ of the system described by Eqn. I.1, I.2, I.3, and I.4 with input $w[k]$ that satisfies $||w||_\infty = 1$. Let $\mathcal{C}[i] = (\mathcal{C}_c[i], \mathcal{C}_d[i])$, where $\mathcal{C}_c[i]$ represents the portion of $\mathcal{C}[i]$ due to the state x, and $\mathcal{C}_d[i]$ represents the portion of $\mathcal{C}[i]$ due to state z. Then the following holds:

$$||\mathcal{C}_c[N]||_2 \leq \max\{||\mathcal{C}_c[0]||_2, B\} \tag{III.9}$$

where $B = (1 + \epsilon^*)\sqrt{(\gamma^*)^{-1} + 1}$.

Proof: To begin, note that any $\mathcal{R}^*-$cycle of this system satisfies the constraint that

$$\mathcal{C}_c[N] = A_1^{M_2} V_j A_1^{M_1} \mathcal{C}_c[0]$$

for $j = 0, 1$ or 2 where V_1 and V_2 are defined as in Proposition III.3, and $V_0 = I_2$, the 2×2 identity matrix. Moreover, $M_1, M_2 \geq 0$, and $A_1^{M_1}\mathcal{C}_c[0] \in \mathcal{R}^*$. The case $j = 0$ corresponds to a trivial case in which the first element of $\mathcal{C}_d[i]$ is constant for $i = 0, 1, \ldots N$ and for which the second element of $\mathcal{C}_d[i]$ (which represents the state variable z_2) is equal to 1 for $i = 0, 1, \ldots N$. In this case we have $\mathcal{C}_c[N] = A_1^N \mathcal{C}_c[0]$.

The cases where $j = 1$ and $j = 2$ correspond to cases where z_2 (and hence the second component of $\mathcal{C}_d[i]$) increments through the cycle. The term $A_1^{M_1}$ accounts for rotation of the state x before z_2 begins to increment. Note in this situation that z_2 must begin incrementing either while the continuous portion of the state lies in the strip $|x_1| \leq 1$ or one time step after the continuous portion of the state leaves this strip. Hence, by choosing V_1 or V_2 as appropriate, we can always choose M_1 such that $A_1^{M_1}\mathcal{C}_c[0] \in \mathcal{R}^*$. Finally, the $A_1^{M_2}$ term represents possible additional rotation after z_2 has become 1 again in order to guarantee that $\mathcal{C}_c[N] \in \mathcal{V}^*$ and, hence $\mathcal{C}[N] \in \mathcal{R}^*$.

For the case when $j = 0$, the statement follows trivially since A_1 is a rotation matrix and, hence, $||\mathcal{C}_c[N]||_2 = ||A_1^N \mathcal{C}_c[0]||_2 = ||\mathcal{C}_c[0]||_2$. When $j = 1$ or $j = 2$, we have that

$$||\mathcal{C}_c[N]||_2 = ||A_1^{M_2} V_j A_1^{M_1} \mathcal{C}_c[0]||_2 = ||V_j A_1^{M_1} \mathcal{C}_c[0]||_2.$$

If we let $x = A_1^{M_1}\mathcal{C}_c[0]$, then $x \in \mathcal{R}^*$, and, hence,

$$||\mathcal{C}_c[N]||_2 = ||V_j x||_2, \quad x \in \mathcal{R}^*$$

for which the given bound immediately follows as a result of Proposition III.4. ■

Proposition III.5 provides a useful bound relating the beginning of an $\mathcal{R}^*-$cycle and the end of the same $\mathcal{R}^*-$cycle. Since, however, we are interested in computing a bound on $||x||_\infty$ which involves providing a bound for *all* times (not just the beginning and end of $\mathcal{R}^*-$cycles), a stronger bound is desired. The bound of Proposition III.5 can be extended by making use of the following basic Lemma, whose proof is immediate and is left to the reader.

Lemma III.2: Consider a linear transformation A which, for a given vector x satisfies $||Ax||_2 \geq ||x||_2$. Then $||A^2x||_2 \geq ||Ax||_2$.

We now prove the following stronger assertion.

Proposition III.6: Consider an $\mathcal{R}^*-$cycle $\mathcal{C}[i], i = 0, 1, \ldots N$ of the system described by Eqn. I.1, I.2, I.3, and I.4 with input $w[k]$ that satisfies $||w||_\infty = 1$. Let $\mathcal{C}[i] = (\mathcal{C}_c[i], \mathcal{C}_d[i])$, where $\mathcal{C}_c[i]$ represents the portion of $\mathcal{C}[i]$ due to the state x, and $\mathcal{C}_d[i]$ represents the portion of $\mathcal{C}[i]$ due to the state z. Then the following holds:

$$\max_{i=0,1,\ldots,N} ||\mathcal{C}_c[i]||_2 \leq \max\{||\mathcal{C}_c[0]||_2, B\} \tag{III.10}$$

where $B = (1 + \epsilon^*)\sqrt{(\gamma^*)^{-1} + 1}$.

Proof: To begin, note that $\mathcal{C}_c[i]$ takes one of the following three forms for any $i = 0, 1, \ldots, N$:

$$\begin{aligned} \mathcal{C}_c[i] &= A_1^{M_1}\mathcal{C}_c[0], \quad M_1 \geq 0 \\ \mathcal{C}_c[i] &= A_2^{M_2} A_1^{M_1}\mathcal{C}_c[0], \quad M_1 \geq 6000, M_2 \leq 1000 \\ \mathcal{C}_c[i] &= A_1^{M_2} A_2^{1000} A_1^{M_1}\mathcal{C}_c[0], \quad M_1 \geq 6000, M_2 \geq 0 \end{aligned}$$

When $\mathcal{C}_c[i]$ is of either the first or third listed forms, the result follows from the analysis in the proof of Proposition III.5. Hence, the desired bound need only be proved when $\mathcal{C}_c[i]$ is of the second listed form.

We consider two cases: one in which $||\mathcal{C}_c[N]||_2 \leq ||\mathcal{C}_c[0]||_2$, and one in which $||\mathcal{C}_c[N]||_2 > ||\mathcal{C}_c[0]||_2, ||\mathcal{C}_c[N]||_2 \leq B$. In the first case, let $x = A_1^{M_1}\mathcal{C}_c[0]$, and note that $||\mathcal{C}_c[0]||_2 = ||x||_2$. We wish to show that $||A_2^{M_2}x||_2 \leq ||x||_2$ for $M_2 \leq 1000$. Suppose that this assertion is not true, i.e. that there exists some $M_2 < 1000$ such that $||A_2^{M_2}x||_2 > ||x||_2$. Then repeated application of Lemma III.2 shows that $||A_2^{1000}x||_2 > ||x||_2$. However,

$$||A_2^{1000}x||_2 = ||\mathcal{C}_c[N]||_2 \leq ||\mathcal{C}_c[0]||_2 = ||x||_2,$$

an obvious contradiction. Hence, it follows that if $||\mathcal{C}_c[N]||_2 \leq ||\mathcal{C}_c[0]||_2$ then $||\mathcal{C}_c[i]||_2 \leq ||\mathcal{C}_c[0]||_2$ for all $i = 0, 1, \ldots N$.

In the second case, again let $x = A_1^{M_1}\mathcal{C}_c[0]$, and note that $||\mathcal{C}_c[N]||_2 = ||A_2^{1000}x||_2 > ||x||_2$. We show now that $||A_2^{M_2}x|| \leq ||A_2^{1000}x||$ for all $0 \leq M_2 \leq 1000$. Assume that there exists $0 \leq M_2 < 1000$ such that $||A_2^{M_2}x||_2 > ||A_2^{1000}x||_2$. Then $||A_2^{M_2}x||_2 > ||x||_2$, and by Lemma III.2, $||A_2^{1000}x||_2 \geq ||A_2^{M_2}x||_2$. From this result, it then follows that if $||\mathcal{C}_c[N]||_2 \leq B$, then $||\mathcal{C}_c[i]||_2 \leq B$ for all $i = 0, 1, \ldots, N$. Combining both the first and second cases, we find that $||\mathcal{C}_c[i]||_2 \leq \max\{||\mathcal{C}_c[0]||_2, B\}$ for all $i = 0, 1, \ldots, N$ as desired. ■

IV. Main Result

Proposition III.6 provides a useful bound on the behavior of the continuous portion of individual cycles. Since, by Thm. II.1, we can view full state trajectories as concatenations of cycles, we can use the boundedness result for individual cycles to now prove the ultimate bound on $||x||_\infty$. The remainder of this section is devoted to that proof.

Proof: [Proof of Proposition I.1] Recall from Thm. II.1 that there exists a finite or infinite sequence of times $\{j_i\}$ which demark the beginnings and ends of cycles. We will first show the following intermediate bound: for each trajectory $(x[k], z[k])$ of the system with input $w[k]$ satisfying $||w||_\infty = 1$,

$$||x[k]||_2 \leq \max\{||x[j_1]||_2, B\}, k \geq j_1.$$

Proposition III.6 proves this assertion for $j_1 \leq k \leq j_2$. Assume that the statement holds for all times up to j_i, i.e.

$$||x[k]||_2 \leq \max\{||x[j_1]||_2, B\}, j_1 \leq k \leq j_i.$$

Now, for $j_i \leq k \leq j_{i+1}$,

$$\begin{aligned} ||x[k]||_2 &\leq \max\{||x[j_i]||_2, B\} \\ &\leq \max\{||x[j_1]||_2, B\} \end{aligned}$$

If the sequence $\{j_i\}$ is infinite, then the given bound holds for all $k \geq j_1$. If the sequence is finite, then there exists some time j_N such that $(x[k], z[k]) \in \mathcal{R}^*$ for all $k \geq j_N$. This, however, implies that $||x[k]||_2 \leq (1 + \epsilon^*)$ for all $k \geq j_N$ (since $z_2[k] = 1$ for all $k \geq j_N$, $||x[k]||$ is constant for $k \geq j_N$, and if $||x[j_N]||_2 > (1 + \epsilon^*)$, $x[k] \notin \mathcal{V}^*$ for some $k \geq j_N$). Since $(1+\epsilon^*) \leq B$, the bound for $k \geq j_1$ follows.

What remains is to find a constant α such that $||x[j_1]||_2 \leq \alpha||x[0]||_2$ for any trajectory of the system with $||w||_\infty = 1$. This can be performed by engaging an analysis similar to that of the proof of Proposition II.2. Without loss of generality, consider the case where the initial condition $(x[0], z[0]) \notin \mathcal{R}^*$. We shall first consider the case where $z[0] \neq 1$. In this case, there exists some $0 < k_1 < 7000$ such that $z_2[k_1] = 7000$ and $x[k_1] = A_2^{1000}A_1^{M_1}x[0]$ for some $M_1 \geq 0$. It follows that

$$||x[k_1]||_2 \leq ||A_2||^{1000}||x[0]||_2 = \exp\left(\frac{\pi}{8}\right)||x[0]||_2.$$

If $x[k_1] \in \mathcal{V}^*$, then we may take $j_1 = k_1$. Otherwise, $z_2[k]$ will remain 1 for $k_1 \leq k \leq k_2$ where $x[k_2]$ satisfies one of the following two conditions:

1) $|x_1[k_2]| \leq 1 + \epsilon^*$
2) $x_1^2[k_2] \leq \gamma^* x_2^2[k_2]$

In either case, $x[k_2] \in \mathcal{V}^*$ and, hence, we may take $j_1 = k_2$ with the bound

$$||x[j_1]||_2 \leq \exp\left(\frac{\pi}{8}\right)||x[0]||_2.$$

If $z_2[0] = 1$ and $x[0] \notin \mathcal{V}^*$, an analysis similar to above shows that the above bound holds in this case as well so that, overall, for any trajectory of the system with $||w||_\infty = 1$,

$$||x[j_1]||_2 \leq \exp\left(\frac{\pi}{8}\right)||x[0]||_2.$$

Now, using the fact that $||x[k]||_2 \geq ||x[k]||_\infty$, we find

$$||x||_\infty \leq \max\{\exp\left(\frac{\pi}{8}\right)||x[0]||_2, (1 + \epsilon^*)\sqrt{(\gamma^*)^{-1} + 1}\}.$$

■

V. Concluding Remarks

While the $\mathcal{R}-$cycles constructed here applied to a very specific case study, preliminary work indicates that for a large class of systems with the structure depicted in Fig. I.1, $\mathcal{R}-$cycles can be used to establish input-to-state stability when either or both inputs $v(t)$ and $w(t)$ are present, and can also be used to establish asymptotic stability of the sampled continuous state $x[k]$ when both $v(t)$ and $w(t)$ are identically 0. Note that while the formal proofs involving $\mathcal{R}-$cycles to prove stability may be somewhat lengthy, the *computational* portion of finding the parameters α and β in the upper bound are not taxing since it only requires solving a simple semi-definite program. Hence, for cases when they are applicable, $\mathcal{R}-$cycles can provide a computationally efficient way of computing a stability measure for a system.

References

[1] B. Hu, G. Zhai and A. N. Michel, Hybrid Output Feedback Stabilization of Two-Dimensional Linear Control Systems, *Proceedings of the 2000 American Control Conference*, pp. 2184-2188, Chicago, Illinois, June, 2000.

[2] S. Huang, M. R. James, D. Nesic and P.M. Dower, Analysis of Input to State Stability for Discrete-time Nonlinear Systems via Dynamic Programming, *Automatica*, vol. 41, No. 12 (2005), 2055-2065.

[3] Z.P. Jiang and Y. Wang, Input-to-state Stability for Discrete-time Nonlinear Systems, *Automatica*, 37, 857-869, 2001.

[4] D. Liberzon and D. Nesic, Input to State Stabilization of Linear Systems with Quantized Feedback, the joint *44th Conf. Decis. Contr.* and *Europ. Contr. Conf.*, Seville, Spain, pp. 8197-8202, 2005.

[5] A. Megretski and A. Rantzer, System Analysis via Integral Quadratic Constraints, *IEEE Transactions on Automatic Control*, 47(6), pp. 819-830, June, 1997.

[6] D. Nesic and D. S. Laila, A Note on Input-to-state Stabilization for Nonlinear Sampled-data Systems", *IEEE Trans. Automat. Contr.*, vol. 47 (2002), pp. 1153-1158.

[7] K. R. Santarelli, A. Megretski, and M. A. Dahleh, On the Stabilizability of Two-dimensional Linear Systems via Switched Output Feedback, *Proceedings of the 2005 American Control Conference*, pp. 3778-3783, Portland, Oregon, June, 2005.

[8] X. Xu and P. J. Antsaklis, Design of Stabilizing Control Laws for Second-order Switched Systems, *Proceedings of the 14th IFAC World Congress,* vol. C., pp. 181-186, Beijing, P.R. China, July 1999.

[9] X. Xu and P. J. Antsaklis, Stabilization of Second-order LTI Switched Systems, *ISIS Technical Report isis-99-001,* Dept. of Elec. Enging., University of Notre Dame, January, 1999.