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Optimal Controller Synthesis for Second Order LTI Plants with Switched Output Feedback

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***Abstract*— We derive a switched output feedback control law for the class of second order linear systems of relative degree two which maximizes a quantity that measures the rate of convergence of the state trajectories to the origin. After providing a formal definition for the rate of convergence and formulating an infinite horizon optimization problem, we explore a corresponding finite horizon problem for the specific case where the plant is a double integrator ($P(s) = 1/s^2$) and use qualitative information about the behavior of the optimal state trajectories to derive a control law for the corresponding infinite horizon problem. We then derive an optimal controller for a general second order plant of relative degree two by relating it to the double integrator case study through appropriate transformations. We conclude with a design example.**

I. Introduction

Stabilization of continuous time systems via hybrid feedback (in which a controller which possesses both continuous and discrete dynamics is employed) is a problem that has received much attention in the recent literature. Artstein first raised this question via examples [1]. Litsyn et. al. show in [3] that the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{I.1}$$

with (A, B) reachable and (C, A) observable can be stabilized via a hybrid feedback controller which uses a countable number of discrete states (and no continuous states). A natural question arises as to whether a hybrid feedback controller can be designed which uses a *finite* number of states instead. Hu et. al. first gave a partial answer to this question for second order systems in [4] based upon the conic switching laws of [7] and [8].

In our prior work [5], we provide necessary and sufficient conditions on the stabilizability of the system described by Eqn. I.1 under the feedback control law $u(x) = v(x)Cx$ when Eqn. I.1 is second order. The main result, repeated here, is as follows:

Theorem I.1: Consider the system I.1 with $A \in \mathbf{R}^{2\times2}$, $B \in \mathbf{R}^{2\times1}$, and $C \in \mathbf{R}^{1\times2}$ where neither C nor B is identically 0. Define the root locus of this system to be the locus of eigenvalues of $A + kBC$ as k varies continuously over $\mathbf{R}$. Then exactly one of the following statements is true:

1) The system is static output feedback stabilizable.
2) The system is not static output feedback stabilizable, but it has root locus which takes on complex values for some values of $k \in \mathbf{R}$ and is stabilizable by a control law $v(x_1, x_2)$ of the following form:

$$v(x_1, x_2) = \begin{cases} k_1 & w'x = 0 \\ k_2 & w'x \neq 0 \end{cases}$$

with $w'q = 0$, where q is the sole stable, real eigenvector of the matrix $A + k_1BC$, and where k_2 is chosen such that the eigenvalues of $A + k_2BC$ are complex.
3) The system has a root locus which is real for all values of $k \in \mathbf{R}$ and is not stabilizable by control of the form $u(x) = v(x)Cx$ for any choice of $v(x_1, x_2)$.

When it is possible, the above result provides a constructive method of designing a stabilizing controller which implements either static or switched output feedback; however, the result is not discriminatory in the sense that, if a given second order system satisfies either the first or second item in Thm. I.1, there are *several* controllers which achieve stability. The goal of the current work is to explore a method of designing controllers which maximize the *rate of convergence* (to be defined formally in the following section) of the state trajectory to the origin.

After we formally define the optimization problem to be considered, we will begin by finding an optimal controller for the specific case when the plant under consideration is a double integrator with transfer function $P(s) = 1/s^2$. We will then find a general controller design for all second order plants of relative degree two by making appropriate transformations to change the problem with a given plant of relative degree two into an optimization problem involving a double integrator and then transforming the control design back into the original state space. We conclude by presenting a design example to illustrate the methodologies described here.

Due to space constraints, several proofs and discussions have been curtailed or omitted. The reader is referred to [6] for a more detailed treatment of the subject matter presented here.

II. Rate of Convergence: Definitions

In this section, we introduce the metric of *rate of convergence* over which our ensuing designs will be optimized. We begin with some definitions:

Definition II.1: The autonomous system described by $\dot{x} = f(x)$ [1] is said to be *globally exponentially stable* if there exist constants $M, \beta > 0$ such that, for all solutions $x(t)$,

$$||x(t)||_2 \leq Me^{-\beta t}||x(0)||_2 \ \forall t \geq 0. \quad \text{(II.2)}$$

Definition II.2: A function $f(x)$ is said to be *homogeneous* if for every $c \in \mathbf{R}$, $f(cx) = cf(x)$.

Definition II.3: For a globally exponentially stable autonomous system of the form $\dot{x} = f(x), x(0) = x_0$ where $f(x)$ is homogeneous and piecewise continuous, we define the *rate of convergence* R as

$$R = \min_{||x_0||=1} \liminf_{T\to\infty} -\frac{1}{2T} \ln\left(||x(T)||^2\right).$$

Def. II.3 finds the *largest* real number β such that all solutions of the differential equation satisfy $||x(t)|| \leq Me^{-\beta t}||x(0)||$ for some $M > 0$. Note that, because of the assumed exponential stability of the system, $R > 0$, since for any initial condition $x(0)$,

$$R \geq \liminf_{T\to\infty} -\frac{1}{2T} \ln\left(M^2 e^{-2\beta T}||x(0)||^2\right) = \beta.$$

While for general nonlinear systems, this definition may not be well-defined (the limit infimum may approach $+\infty$ or the minimization over the unit circle may not capture the behavior of all solutions), the assumptions of homogeneity and piecewise continuity ensure that the definition of R is a sensible one. The reader is referred to [6] for a discussion of this.

A useful property about the rate of convergence that we will utilize in our optimization study is the following:

Corollary 1: Define the $P-$*rate of convergence* R_P as

$$\min_{||x_0||=1} \liminf_{T\to\infty} -\frac{1}{2T} \ln\left(x(T)'Px(T)\right)$$

where $P = P' > 0$. Then $R_P = R_I = R$.

Proof: See [6] ∎

III. Problem Formulation

Now that we have formally defined our optimization metric, we can formulate the problem under investigation. Consider a single-input, single-output linear system of the form Eqn. I.1 where the corresponding transfer function $C(sI - A)^{-1}B$ is second order and of relative degree two. Further consider a feedback control law of the form $u(x) = v(x)Cx$ where $u(x)$ is homogeneous so that the overall interconnected system is an autonomous system which takes the form

$$\dot{x} = Ax + v(x)BCx, \quad x(0) = x_0\text{: given.} \quad \text{(III.3)}$$

[1] We assume throughout this paper that all vector fields are defined such that a unique solution exists for every initial condition $x(0)$.

Here, the scalar function $v(x)$ lies in a set $V(v_0)$, where $V(v_0)$ is defined as the set of all $v(x)$which are bounded and satisfy $v(x) \in [-v_0 + \gamma, v_0 + \gamma] \ \forall x \in \mathbf{R}^2$, with γ an appropriately chosen constant and where $v_0 > 0$ is large enough to satisfy the following conditions:

- There exists v_1 with $v_1 \in [-v_0+\gamma, v_0+\gamma]$ such that the eigenvalues of $A + v_1BC$ form a complex conjugate pair.
- There exists v_2 with $v_2 \in [-v_0+\gamma, v_0+\gamma]$ such that at least one of the eigenvalues of $A + v_2BC$ lies strictly in the open left half plane.

In addition to the above, the assumed homogeneity of $u(x)$ implies that $v(x)$ satisfies the following property:

Proposition III.1: If $u(x)$ is homogeneous and $u(x)$ and $v(x)$ are related by the transformation $u(x) = v(x)Cx$, then $v(x) = v(\alpha x)$ for all x with $Cx \neq 0$ and for all $\alpha \neq 0$.

Proof: For any $\alpha \neq 0$

$$\begin{aligned} u(\alpha x) &= \alpha v(\alpha x)Cx \\ \alpha u(x) &= \alpha v(x)Cx \end{aligned}$$

Since $u(\alpha x) = \alpha u(x)$, equating the two expressions and dividing by αCx yields the result. ∎

For simplicity, we will examine choices of $v(x)$ for which $v(\alpha x) = v(x)$ for *all* x as this will not affect our choice of an optimal controller.

It is easily verified that any choice of $v(x)$ that satisfies the bulleted criteria above will admit a stabilizing controller as described by Thm. I.1. In our main result, we will pose a condition equivalent to the above two constraints that is easily verified by checking a simple condition on the value of v_0 and the parameters of the transfer function $P(s)$.

The "offset" parameter γ given above is a function of the parameters of the corresponding transfer function $C(sI - A)^{-1}B$. An exact selection of γ for each given transfer function will be constructed when we present the main result.

It is clear that, for each choice of $v(x)$, the autonomous system Eqn. III.3 has an associated rate of convergence R. For a given plant Eqn. I.1 and given value of $v_0 > 0$, the task at hand, then, is to find a choice of $v(x) \in V(v_0)$ such that the corresponding rate R is maximum. Note that the optimal R is implicitly a function of v_0 and, hence, we use the notation

$$R^*(v_0) = \max_{v(x)\in V(v_0)} R(v(x))$$

to denote this optimal value.

In the sections that follow, we will find a choice of $v^*(x)$ which achieves the maximal rate $R^*(v_0)$ in the above optimization problem and will also explicitly characterize the optimal value $R^*(v_0)$ in terms of v_0 and the parameters of the transfer function $C(sI - A)^{-1}B$. We will prove optimality of the resulting controllers by first finding an optimal controller for the specific case in which the plant under consideration is a double integrator. We will then find optimal controllers/rates for the problem in which the plant is a general second order system of relative degree two by using appropriate transformations to relate the optimal controller and rate for a given plant to the optimal controller and rate of a double integrator.

IV. Optimal Control of a Double Integrator

In this section, we synthesize the design of an optimal controller for a plant which acts as a double integrator, $P(s) = 1/s^2$, with canonical state-space description

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{(IV.4)} \\ y &= x_1 \quad \text{(IV.5)} \end{aligned}$$

which, under the feedback law $u(x) = v(x)y$, yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ v(x) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad \text{(IV.6)}$$

We consider here the task of finding the minimal rate $R^*(1)$ when γ provided in the description of $V(v_0)$ of the previous section is equal to 0 and when v_0 is equal to 1. That is, we wish to find $v^*(x)$ with $|v^*(x)| \leq 1 \; \forall x \in \mathbf{R}^2$ [2] such that the rate

$$R(v^*(x)) = \min_{||x(0)||=1} \liminf_{T\to\infty} -\frac{1}{2T} \ln\left(||x^*(T)||^2\right),$$

is as large as possible, where $x^*(t)$ denotes a solution to Eqn. III.3 with $v(x) = v^*(x)$, i.e.,

$$R(v^*(x)) \geq R(v(x)) \; \forall v(x) \in V(1).$$

A. Finite Horizon Optimal Control

In order to make headway into solving the above problem, we will consider the following relaxed finite horizon problem: for a given horizon $T > 0$, define the set W as

$$W = \{w(t) : [0, T] \to \mathbf{R} : |w(t)| \leq 1\}.$$

We are interested in finding a choice of $w(t) \in W$, which we will refer to as $w^*(t)$, such that the corresponding solution $x^*(t)$ of the system equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ w(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \text{(IV.7)}$$

is as small as possible, i.e.

$$||x^*(T)|| \leq ||x(T)|| \qquad \forall w(t) \in W$$

where $x(t)$ denotes the solution corresponding to a particular choice of $w(t)$.

A few comments are in order. First, it is clear from the definition of R in Def. II.3 that choosing $v(x) \in V(1)$ to minimize $||x(T)||^2$ for each fixed T is equivalent to maximizing $R(1)$. After all, if $x^*(T)$ achieves the minimal norm for a given initial condition $x(0) = x_0$, $||x^*(T)|| \leq ||x(T)||$ for all solutions with $v(x) \in V(1), x(0) = x_0$. Hence,

$$-\frac{1}{2T} \ln\left(||x(T)||^2\right) \leq -\frac{1}{2T} \ln\left(||x^*(T)||^2\right)$$

and the statement follows by noting that the minimization and limit inferior operations preserve the above inequality.

Also, note that the specific choice of initial condition $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}'$ is artificial and is chosen for simplicity. Once we have solved the given finite horizon problem for this specific initial condition, we will be able to construct optimal solutions for several other initial conditions which, in the end, will lead us to a design for $v(x)$ in the original optimization problem.

Finally, we note that the given optimization over the set W provides a lower bound for the minimum achievable value of $||x(T)||$ for the original feedback law $u(x) = v(x)y$. To see this, for a given initial condition $x(0) = x_0$, define $\tilde{W}$ as

$$\begin{aligned} \tilde{W} \;\; = \;\; & \{\tilde{w}(t) : [0, T] \to \mathbf{R} : \tilde{w}(t) = v(x(t)), v(x) \in V(1), \\ & \dot{x} = Ax + v(x)BCx, x(0) = x_0\}. \end{aligned}$$

It is clear that $\tilde{W} \subset W$, and, hence,

$$\min_{\tilde{w}\in\tilde{W}} ||x(T)|| \geq \min_{w\in W} ||x(T)||.$$

At this point, the reader may wonder why we are even considering this finite horizon optimization with different structure than our original infinite horizon optimization. Our reasoning is twofold. First, by examining the qualitative behavior of the optimal solutions to this finite horizon problem, we will be able to invent a form of an optimal controller $v(x)$ to the original infinite horizon problem which we will, then, be able to prove is optimal. Second, the setup for the finite horizon problem fits the framework of the celebrated *Pontryagin Minimum Principle*. The essential gist of our reasoning, then, is that we solve this finite horizon problem, for which efficient tools exist to obtain an optimal solution, so as to gain intuition into the way we should design $v(x)$ in the infinite horizon case.

The above optimization problem can be solved efficiently using Pontryagin's Minimum Principle [2]. As the exact technicalities of utilizing the Minimum Principle to obtain the optimal choice of $w(t)$ is not the main focus of our exposition, the interested reader is referred to [6] for a complete description of how we use the Minimum Principle to generate the optimal choice of $w(t)$. The end result of the analysis shows that the optimal choice of $w(t)$ is a piecewise-constant "bang-bang" controller (i.e., $w(t) \in \{-1, 1\}$ for all t). Moreover, examination of the geometric behavior of the phase portraits for the optimal $w(t)$ as the horizon time $T \to \infty$ allows us to hypothesize a form of an optimal controller in the original infinite horizon problem.

B. Extension to the Infinite Horizon Problem

By examining the geometric behavior of the optimal trajectories of the finite horizon problem of the prior section,

[2]It can be easily verified the specific choice of $v_0 = 1$ and $\gamma = 0$ satisfy the two conditions listed in the description of $V(v_0)$ of the previous section for this plant.

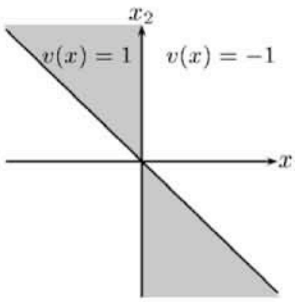

Fig. IV.1. Switching law of Eqn. IV.8

we are able to hypothesize that the function $v^*(x)$ given by

$$v^*(x) = \begin{cases} 1 & x_1(x_1 + x_2) \leq 0 \\ -1 & x_1(x_1 + x_2) > 0 \end{cases} \qquad \text{(IV.8)}$$

which is illustrated graphically in Fig. IV.1 will maximize $R(1)$ for the original infinite horizon optimization problem described at the beginning of this section. As it turns out, this feedback law *does* maximize $R(1)$, a property which we now show in two steps, one which establishes an upper bound for $R^*(1)$ and the other which shows that $v^*(x)$ of Eqn. IV.8 achieves this bound.

Proposition IV.2: For the double integrator of Eqn. IV.4 and IV.5 with some choice of $|v(x)| \leq 1 \; \forall x \in \mathbf{R}^2$,

$$R^*(1) \leq 1.$$

Proof: Using the notation

$$A(v) = \begin{bmatrix} 0 & 1 \\ v & 0 \end{bmatrix}, \qquad \text{(IV.9)}$$

we have

$$\begin{aligned} \frac{d}{dt}||x(t)||^2 &= x(t)'(A(v(x(t))) + A'(v(x(t))))x(t) \\ &\geq \min_{|v|\leq 1} \lambda_{\min}(A(v) + A'(v))||x(t)||^2 \\ &= -2||x(t)||^2 \end{aligned}$$

Hence,

$$||x(t)||^2 \geq e^{-2t}||x(0)||^2 \triangleq r(t).$$

Now, for any initial condition $x(0)$,

$$\begin{aligned} \liminf_{T\to\infty} -\frac{1}{2T} \ln\left(||x(T)||^2\right) &\leq \lim_{T\to\infty} -\frac{1}{2T}\ln(r(T)) \\ &= 1 \end{aligned}$$

from which it immediately follows that $R^*(1) \leq 1$. ■

Proposition IV.3: For the double integrator with feedback law $u(x) = v^*(x)y$ with $v^*(x)$ as in Eqn. IV.8, the rate of convergence R is equal to 1.

Proof: The proof, present in [6], shows that the above claim is true by explicitly computing the solution for each initial condition $x(0)$ and by showing that the limit inferior is always equal to 1. ■

V. Optimal Control of Second Order Systems of Relative Degree Two

In this section, we generalize the result of the previous section to all second order linear systems with a transfer function of the form

$$P(s) = \frac{c}{s^2 + as + b} \qquad \text{(V.10)}$$

for some $a, b, c \in \mathbf{R}$. In particular, we will choose a specific value for the parameter γ as a function of the parameters of the transfer function $P(s)$ to define the set

$$V(v_0) = \{v(x) : v(x) \in [-v_0 + \gamma, v_0 + \gamma]\}$$

and will then obtain explicit expressions for the optimal rate of convergence $R^*(v_0)$ along with an optimal controller $v^*(x)$. We will first find an optimal controller $v^*(x)$ for a particular state-space realization of the transfer function $P(s)$ when $c > 0$; optimal controllers for all other state-space realizations and for the case when $c < 0$ will be derived via appropriate transformations.

The formal statement that we will first prove is the following:

Proposition V.4: Consider a linear system with transfer function $P(s)$ with $c > 0$ given by Eqn. V.10 with state-space description

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{a}{2} & \sqrt{cv_0} \\ -\frac{\gamma\sqrt{c}}{\sqrt{v_0}} & -\frac{a}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \frac{\sqrt{c}}{\sqrt{v_0}} x_1 \end{aligned}$$

where $\gamma = (4b - a^2)/4c$. Then, if $\sqrt{cv_0} > -a/2$, the control law

$$v^*(x) = \begin{cases} v_0 + \gamma & x_1(x_1 + x_2) \leq 0 \\ -v_0 + \gamma & x_1(x_1 + x_2) > 0 \end{cases} \qquad \text{(V.11)}$$

makes the system $\dot{x} = Ax + v(x)BCx$ globally exponentially stable. Furthermore, the given choice of $v^*(x)$ maximizes the rate of convergence $R(v_0)$ subject to the constraints that $v(x) \in [-v_0 + \gamma, v_0 + \gamma] \; \forall x \in \mathbf{R}^2$, $v(x)$ piecewise continuous, and $v(\alpha x) = v(x)$ for all $x \in \mathbf{R}^2$, $\alpha \neq 0$. Moreover, the maximum value $R^*(v_0)$ is given by

$$R^*(v_0) = \sqrt{cv_0} + \frac{a}{2}.$$

Before establishing this proposition, we need the result of the following Lemma.

Lemma V.1: Consider an exponentially stable system described by

$$\dot{x} = Ax + v(x)BCx$$

where $v(x) \in [-v_0 + \gamma, v_0 + \gamma]$ for an appropriate choice of the parameter γ and for some value of v_0, and where $v(x)$ is piecewise continuous and satisfies the constraint $v(\alpha x) = v(x)$ for all x, $\alpha \neq 0$. Suppose further that this system has corresponding rate of convergence R. Then the following statements are true:

1) The system
$$\dot{z} = (A + \delta I)z + v(z)BCz$$
is exponentially stable for $\delta < R$ and has corresponding rate of convergence $R' = R - \delta$.
2) The system
$$\dot{z} = \mu Az + \mu v(z)BCz$$
is exponentially stable for $\mu > 0$ and has corresponding rate of convergence $R' = \mu R$.

Proof: See [6] ■

Proof: [Proof of Proposition V.4]

We begin by first making the substitution $v(x) = \frac{1}{\sqrt{c}}v'(x) + \gamma$. In terms of the new control $v'(x)$, the problem reduces to finding $v'(x)$ such that the system
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{a}{2} & \sqrt{cv_0} \\ \frac{1}{\sqrt{v_0}}v'(x) & -\frac{a}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(V.12)}$$
is exponentially stable and that the corresponding rate of convergence is maximized subject to the constraint $|v'(x)| \le v_0'$, where $v_0' = \sqrt{c}v_0$. Now, if this system is exponentially stable and has rate $R < a/2$, then the system
$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{cv_0} \\ \frac{1}{\sqrt{v_0}}v'(x) & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{(V.13)}$$
is exponentially stable and has rate $R' = R - a/2$ by item 1 of Lemma V.1. Moreover, the system
$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ v''(x) & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{(V.14)}$$
where $v''(x) = v'(x)/\sqrt{c}v_0$, $|v''(x)| \le 1$ for all x, is exponentially stable with rate
$$R'' = \frac{1}{\sqrt{cv_0}}\left(R - \frac{a}{2}\right)$$
by item 2 of Lemma V.1 But this problem is exactly the double integrator problem of the previous section for which we have already found an optimal controller given by
$$v''^*(x) = \begin{cases} 1 & x_1(x_1 + x_2) \le 0 \\ -1 & x_1(x_1 + x_2) > 0 \end{cases}$$
with an optimal rate $R''^*(1) = 1$. It follows, therefore, that the system described by w is exponentially stable for the control law $v'^*(x) = \sqrt{c}v_0 v''^*(x)$ and has rate $R' = \sqrt{cv_0}$. Now, if $\sqrt{cv_0} > -a/2$, it follows that the original problem is exponentially stable with rate $R = \sqrt{cv_0} + \frac{a}{2}$ with corresponding control law
$$v^*(x) = \begin{cases} v_0 + \gamma & x_1(x_1 + x_2) \le 0 \\ -v_0 + \gamma & x_1(x_1 + x_2) > 0 \end{cases}.$$
That $R = \sqrt{cv_0} + \frac{a}{2}$ is indeed the optimal rate of convergence for the original problem, $R^*(v_0)$, follows from the fact that, for each fixed $v(x)$, R is an affine transformation of the rate R'', i.e., $R = \alpha R'' + \beta$ for some $\alpha, \beta \in \mathbf{R}$. Hence, the maximum value of the left-hand side R^* is equal to the $\alpha R''^* + \beta$ where R''^* denotes the maximum value of R'', and, thus, the controller of Eqn. V.11 is an optimal controller with optimal rate $R^*(v_0) = \sqrt{cv_0} + a/2$. ■

To obtain an optimal design for all other state-space realizations of a given second order transfer function of relative degree two, essentially, one need only apply a simple change of coordinates:

Proposition V.5: Consider an exponentially stable system of the form $\dot{x} = Ax + v(x)BCx$ with rate R where $v(x)$ takes the form
$$v(x) = \begin{cases} v_1 & x'(F_1F_2' + F_2F_1')x \le 0 \\ v_2 & x'(F_1F_2' + F_2F_1')x > 0 \end{cases}$$
where F_1, F_2 are column vectors of appropriate dimension. Then the system $\dot{z} = \tilde{A}z + \tilde{v}(z)\tilde{B}\tilde{C}z$ with $\tilde{v}(z)$ given by
$$\tilde{v}(z) = \begin{cases} v_1 & z'(\tilde{F}_1\tilde{F}_2' + \tilde{F}_2\tilde{F}_1')z \le 0 \\ v_2 & z'(\tilde{F}_1\tilde{F}_2' + \tilde{F}_2\tilde{F}_1')z > 0 \end{cases}$$
with
$$\tilde{A} = T^{-1}AT,\ \tilde{B} = T^{-1}B,\ \tilde{C} = CT,\ \tilde{F}_i = T'F_i,\ i = 1,2$$
where T is an invertible matrix is also exponentially stable with rate R.

Proof: Performing the change of coordinates $x = Tz$ shows that the above system defined by z has P-rate of convergence equal to R with $P = T'T$. But Cor. 1 implies that the rate of convergence of the new system defined by z is equal to R, as well. ■

The optimal control law $v^*(x)$ of Eqn. V.11 takes the form listed in Prop. V.5 where we take
$$F_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Hence, if an arbitrary state-space description for a plant $P(s)$ is related to the state-space description given in Prop. V.4 by the coordinate transformation $x = Tz$, then we obtain an optimal control law $v^*(z)$ by finding the vectors $\tilde{F}_1$ and $\tilde{F}_2$ in Prop. V.5. That the control law $v^*(z)$ is indeed optimal is a simple fact whose proof is left to the reader.

For the case when $c < 0$, we obtain an optimal controller by solving the optimization problem for $c' = -c$ and then inverting the sign of the feedback, i.e., by setting $v^*(x) = -v'^*(x)$, where $v'^*(x)$ is the optimal solution when c is replaced by c'. The reader is referred to [6] for a more complete description.

VI. Design Example

In this section, we show how to use the result of Prop. V.4 in the context of a specific example.

Example VI.1 Consider the unstable LTI plant
$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -12 & 7 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= z_1. \end{aligned}$$
with transfer function
$$P(s) = \frac{1}{s^2 - 7s + 12}.$$

For this particular plant, note that γ of Prop. V.4 is $\gamma = -1/4$. The task at hand is to find a choice of $v(z) \in [-v_0 - 1/4, v_0 - 1/4]$ such that, under the control law $u(z) = v(z)y$, the corresponding rate of convergence is maximized when $v_0 = 100$. Note that Prop. V.4 tells us that we can find a controller $v(z) \in [-v_0 - 1/4, v_0 - 1/4]$ that yields and exponentially stable interconnection so long as $\sqrt{v_0} > 7/2$. Hence, for $v_0 = 100$, the previous condition is satisfied, and we may now go about the business of finding an optimal controller.

We know that

$$v^*(x) = \begin{cases} 99.75 & x_1(x_1 + x_2) \le 0 \\ -100.25 & x_1(x_1 + x_2) > 0 \end{cases}$$

is an optimal controller with rate $R^*(100) = 6.5$ for the state-space description

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 3.5 & 10 \\ 0.025 & 3.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= 0.1x_1. \end{aligned}$$

By diagonalizing the "A" matrix for both state-space descriptions, a simple calculation shows that the two descriptions are related via the coordinate transformation $x = Tz$ where

$$T = \begin{bmatrix} 10 & 0 \\ -3.5 & 1 \end{bmatrix}.$$

Using this transformation to compute $\tilde{F}_i$, $i = 1, 2$ of Prop. V.5, we establish the following optimal control law in terms of the original state-space description:

$$v^*(z) = \begin{cases} 99.75 & x_1(6.5x_1 + x_2) \le 0 \\ -100.25 & x_1(6.5x_1 + x_2) > 0 \end{cases} \qquad \text{(VI.15)}$$

which is depicted graphically along with a sample trajectory in Fig. VI.2. Notice that one of the boundaries of the cone in which $v(x) = 99.75$ is the stable eigenvector of the matrix

$$\begin{bmatrix} 0 & 1 \\ 88.75 & 7 \end{bmatrix}$$

and that the state trajectory follows this eigenvector for large time. Notice also that the corresponding stable eigenvalue of the above matrix is -6.5 which, as we computed earlier, is our optimal rate $R^*(100)$.

VII. Concluding Remarks

It should be noted that, for a given plant and control bound v_0, the controller which achieves maximum convergence rate $R^*(v_0)$ is *not* unique. Indeed, there are several controllers, including controllers of the form originally listed in [5] which achieve maximum rate. Nevertheless, the work described here does offer several advantages. First, it provides a methodic way of finding *a* controller which achieves minimum rate of convergence. Moreover, ongoing work indicates that the controllers of the form listed here possess useful properties in an application setting for a certain class of plants whereby the controllers here can outperform what can be achieved via *any* LTI controller. Also, to compare the controllers listed here to the controllers listed in our previous work, the controllers given in [5] are clearly non-robust with respect to time delays, whereas implementations of the controllers listed here can be made to be robust with respect to time delays (see [6] for a brief discussion of this).

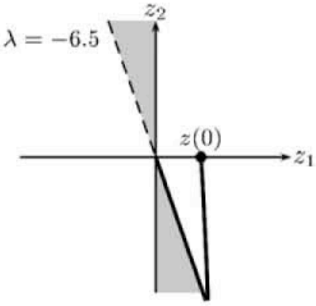

Fig. VI.2. Illustration of the optimal control law $v^*(z)$ of Eqn. VI.15 along with a sample trajectory $z(t)$.

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