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Model Reduction of Discrete-Time Markov Jump linear systems

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***Abstract*— This paper proposes a model reduction algorithm for discrete-time, markov jump linear systems. The main point of the reduction method is the formulation of two generalized dissipation inequalities that in conjunction with a suitably defined storage function enable the derivation of reduced order models that come with a provable a priori upper bound on the stochastic L_2 gain of the approximation error.**

I. Preliminaries

A. System Model

Consider the discrete-time Markov jump linear system (MJLS) $\mathcal{G}$ that has the following state space realization:

$$\begin{aligned} x_{k+1} &= A_{\theta_k} x_k + B_{\theta_k} f_k, \\ y_k &= C_{\theta_k} x_k, \qquad k \in \mathbb{Z}_+, \end{aligned}$$

where the state variable is $x_k \in \mathbb{R}^n$, the input is $f_k \in \mathbb{R}^m$, the parametric input is $\theta_k \in \Theta = \{1, \ldots, N\}$ and the output is $y_k \in \mathbb{R}^p$. Reduced order model candidates are denoted by $\hat{\mathcal{G}}$ and it is required that they lie in the same class of MJLS systems, having the state space realization

$$\begin{aligned} \hat{x}_{k+1} &= \hat{A}_{\theta_k} \hat{x}_k + \hat{B}_{\theta_k} f_k, \\ \hat{y}_k &= \hat{C}_{\theta_k} \hat{x}_k, \qquad k \in \mathbb{Z}_+, \end{aligned}$$

where $\hat{x}_k \in \mathbb{R}^{\hat{n}}$ and $\hat{n} < n$.

In order to quantify the fidelity of $\hat{\mathcal{G}}$, an error system $\mathcal{E}$ is introduced, whose inputs are the common inputs f_k, θ_k of $\mathcal{G}$ and $\hat{\mathcal{G}}$ and whose output is the difference of their outputs, namely $e_k = y_k - \hat{y}_k$.

The parametric input θ_k represents the state of a Markov chain that takes values in a finite set $\Theta = \{1, \ldots, N\}$. The transition probability matrix of the Markov chain is denoted by Q and

$$\mathbf{P}(\Theta_{k+1} = j | \Theta_k = i) = q_{ij} \;\; i, j \;\in \{1, \ldots, N\}, \quad k \in \mathbb{Z}_+.$$

When needed, random variables are denoted by capital letters in order to avoid confusion. The input sequence $\{f_k\}$ is taken to be deterministic and the sequence $\{\Theta_k\}$ has the property that $\forall k \in \mathbb{Z}_+$ each Θ_k is independent of the state history $\{X_0, \ldots, X_k\}$ up to that point.

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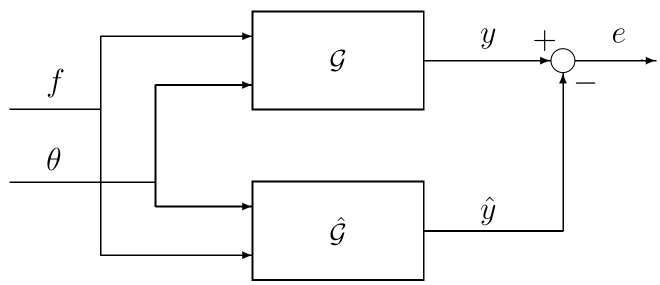

Fig. 1. Error System $\mathcal{E}$

B. Sensitivity measure and stability

Let $l_2^m(\mathbb{Z}_+)$ denote the space of all vector-valued real sequences on nonnegative integers, of dimension m, i.e., $f = \{f_0, f_1, \ldots\}$ with $f_k \in \mathbb{R}^m$, such that

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |f_k|^2 < \infty$$

Here $|f_k|^2 = f_k' f_k$ stands for the square of the Euclidean norm on the underlying vector space. The unit sphere in $l_2^m(\mathbb{Z}_+)$ is denoted by $S_2^m = \{f \in l_2^m(\mathbb{Z}_+) \;:\; \|f\|_2 = 1\}$.

Definition 1.1: The *stochastic L_2 gain* of the system $\mathcal{G}$ is denoted by $\gamma_{\mathcal{G}}$ and is defined for $x_0 = 0$ by

$$\gamma_{\mathcal{G}}^2 = \sup_{f \in S_2^m} \mathbf{E}[\sum_{k=0}^{\infty} |Y_k|^2]$$

Definition 1.2: The system $\mathcal{G}$ with $f_k = 0, \forall k \in \mathbb{Z}_+$ is called *mean square stable*, if for every set of initial conditions $x_0 \in \mathbb{R}^n, \theta_0 \in \Theta$, the second moment of the magnitude of the state converges to zero

$$\mathbf{E}[|X_k|^2] \to 0 \;\text{ as }\; k \to \infty.$$

Definition 1.3: The system $\mathcal{G}$ with $f_k = 0, \forall k \in \mathbb{Z}_+$ is called *exponentially mean square stable* if for every set of initial conditions $x_0 \in \mathbb{R}^n$ and $\theta_0 \in \Theta$, there exist real constants $\beta \geq 1$ and $\rho \in (0, 1)$ such that

$$\mathbf{E}[|X_k|^2] \leq \beta \rho^k |x_0|^2, \quad k \in \mathbb{Z}_+$$

Theorem 1.1: [10] The following statements are equivalent :

- (a) System $\mathcal{G}$ is mean square stable.
- (b) System $\mathcal{G}$ is exponentially mean square stable.

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- (c) For every positive definite matrix $S > S, Q \in \mathbb{R}^{n\times n}$, there exists a unique positive definite matrix $P > 0, P \in \mathbb{R}^{n\times n}$ such that :

$$P - \sum_{i=1}^{N} q_{ij} A_i' P A_i = S,\ j \in \Theta$$

A proof of the above theorem can be found in [10].

Lemma 1.1: Given a system $\mathcal{G}$, if there exists a set of positive definite matrices $(P_1, \ldots, P_N)$ such that the corresponding quadratic functions $V_i(x) = x' P_i x,\ i \in \Theta$ satisfy :

$$\gamma^2 |f|^2 + V_j(x) \geq \sum_{i=1}^{N} q_{ij} [|C_i x|^2 + V(A_i x + B_i f)], \quad (1)$$
$$\forall x \in \mathbb{R}^n, \forall f \in \mathbb{R}^m$$

then the stochastic L_2 gain of $\mathcal{G}$ does not exceed $\gamma \geq 0$.

Proof. The above relation implies

$$\gamma^2 |f_k|^2 + \mathbf{E}[V(X_k)] \geq \mathbf{E}[|Y_k|^2] + \mathbf{E}[V(X_{k+1})], \quad (2)$$
$$\forall f_k \in \mathbb{R}^m, \forall k \in \mathbb{Z}_+.$$

According to definition 1.1 set $x_0 = 0$ and sum relation (2) from $k = 0$ to $k = T$ obtaining

$$\mathbf{E}[\sum_{k=0}^{T} |Y_k|^2] \leq \gamma^2 \sum_{k=0}^{T} |f_k|^2 - \mathbf{E}[V(X_{T+1})].$$

Since V is a nonnegative valued map, $\mathbf{E}[V(X_{T+1})] \geq 0$ thus

$$\mathbf{E}[\sum_{k=0}^{T} |Y_k|^2] \leq \gamma^2 \sum_{k=0}^{T} |f_k|^2.$$

Restricting the input signal f to be on the unit sphere S_2^m gives

$$\mathbf{E}[\sum_{k=0}^{\infty} |Y_k|^2] \leq \gamma^2\ \forall f \in S_2^m$$

and in particular $\gamma_{\mathcal{G}}^2 \leq \gamma^2$ completing the proof. ∎

Theorem 1.2: If the system $\mathcal{G}$ is mean square stable, then its stochastic L_2 gain is finite.

Proof. Let $Q > 0$ be an arbitrary positive definite matrix. Mean square stability guarantees the existence of a positive definite matrix $P > 0$, such that

$$\sum_{i=1}^{N} q_i A_i' P A_i - P = -Q < 0. \quad (3)$$

Define $V(x) = x' \alpha P x$ to be a quadratic function of the state, where $P > 0$ and $\alpha \geq 1$. Using the state equations one obtains the following relation, that is equivalent to condition (1)

$$\begin{bmatrix} x' & f' \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} \leq 0 \ \ \forall x \in \mathbb{R}^n, f \in \mathbb{R}^m \quad (4)$$

where

$$\begin{aligned} W_{11} &= \sum_{i=1}^{N} q_i (A_i' \alpha P A_i + C_i' C_i) - \alpha P \\ W_{12} &= \sum_{i=1}^{N} q_i A_i' \alpha P B_i \\ W_{21} &= Q_{12}' \\ W_{22} &= \sum_{i=1}^{N} q_i B_i' \alpha P B_i - \gamma^2 I \end{aligned}$$

Using the Schur complement idea one can conclude, that a sufficient set of conditions for (4) to hold is

$$W_{11} < 0 \quad (5)$$
$$W_{22} < W_{21} W_{11}^{-1} W_{12} \quad (6)$$

Using (3), relation (5) can be rewritten as

$$\sum_{i=1}^{N} q_i C_i'\ C_i - \alpha Q < 0$$

and there is always an α large enough so that it is satisfied. Setting $F_1 = \sum_{i=1}^{N} q_i B_i' \alpha P B_i$ and $F_2 = W_{21} W_{11}^{-1} W_{12}$ one can rewrite (6) as

$$F_1 - F_2 < \gamma^2 I.$$

The above condition can always be satisfied by taking γ large enough. Thus, there exists an $\alpha \geq 1$ and a $\gamma > 0$ such that $V(x) = x' \alpha P x$ satisfies the dissipation inequality (1). Invoking lemma 1.1 leads to finiteness of the stochastic L_2 gain of $\mathcal{G}$. ∎

A standing assumption in this work is that the process of imposing the aforementioned statistics to the parametric input leads to a mean square stable system.

II. Generalized dissipation inequalities and truncation of states

A. Generalized dissipation inequalities

The model reduction procedure developed in this work relies on the computation of $P > 0$, $\hat{Q} > 0$ for a given mean square stable system $\mathcal{G}$ such that the following set of dissipation inequalities is satisfied:

$$|x|_P^2 \geq \sum_{i=1}^{N} q_i (|A_i x|_P^2 + |C_i x|^2), \quad (7)$$
$$\forall x \in \mathbb{R}^n,$$

$$|x|_{\hat{Q}}^2 + |f|^2 \geq \sum_{i=1}^{N} q_i (|A_i x + B_i f|_{\hat{Q}}^2), \quad (8)$$
$$\forall x \in \mathbb{R}^n, \forall f \in \mathbb{R}^m$$

In the above relations the notation $|z|_P^2 = z' P z$ is used. There is a natural interpretation of (7), (8) in the case where $N = 1$, so that $\mathcal{G}$ reduces to an LTI system. If the system matrices $\{A, B, C\}$ constitute a minimal realization of $\mathcal{G}$, then equation (7) is satisfied with equality using $P = W_o$, and (8) is satisfied with equality using $\hat{Q} = W_c^{-1}$, where W_o, W_c are the observability and controllability Gramians of the system respectively. In the case where $N > 1$, the following two lemmas provide interpretations for P and $\hat{Q}$.

Lemma 2.1: Let $T \in \mathbb{Z}_+$ and consider the unforced $\{f_0, \ldots, f_T\} = \{0, \ldots, 0\}$ response of $\mathcal{G}$ to the initial condition $x_0 \in \mathbb{R}^n$. For an arbitrary $T_0 \in \mathbb{Z}_+$, such that $T_0 < T$ one has

$$\sum_{k=T_0}^{T} \mathbf{E}[|Y_k|^2] \leq \mathbf{E}[|X_{T_0}|_P^2].$$

Proof. The dissipation inequality (7) implies in the unforced case

$$\mathbf{E}[|X_{k+1}|_P^2] + \mathbf{E}[|Y_k|^2] \leq \mathbf{E}|X_k|_P^2,$$

Sum the above relation from $k = T_0$ to $k = T$ to obtain

$$\mathbf{E}[|X_{T+1}|_P^2] + \sum_{k=T_0}^{T} \mathbf{E}[|Y_k|^2] \leq \mathbf{E}[|X_{T_0}|_P^2].$$

Then, noticing that $\mathbf{E}[|X_{T+1}|_P^2] \geq 0$ leads to the desired result. ∎

Lemma 2.2: Let $T \in \mathbb{Z}_+$ and consider the evolution of $\mathcal{G}$ that starts at rest $x_0 = 0$. Then, for an arbitrary input sequence $\{f_0, \ldots, f_T\}$ one has

$$\sum_{k=0}^{T} |f_k|^2 \geq \mathbf{E}[|X_{T+1}|_{\hat{Q}}^2], \ \forall f_k \in \mathbb{R}^m, k \in \{1 \ldots T\}$$

Proof. The dissipation inequality (8) gives in this case

$$\mathbf{E}[|X_{k+1}|_{\hat{Q}}^2] \leq \mathbf{E}[|X_k|_{\hat{Q}}^2] + |f_k|^2, \ \forall f_k \in \mathbb{R}^m.$$

Sum the above relation from $k = 0$ to $k = T$ and note that $x_0 = 0$ to obtain the desired result. ∎

B. Reduction by state truncation

This a brief review of the concept of model reduction by means of state truncation for linear parameter-varying systems. One starts out with the state space representation of $\mathcal{G}$

$$\begin{aligned} x_{k+1} &= A_{\theta_k} x_k + B_{\theta_k} f_k, \\ y_k &= C_{\theta_k} x_k, \qquad k \in \mathbb{Z}_+, \end{aligned} \tag{9}$$

and applies an invertible coordinate transformation $x_k = T\tilde{x}_k$ that puts the "most important" states in first components of the transformed state vector $\tilde{x}_k$. This transformation gives a new state space representation of $\mathcal{G}$

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}_{\theta_k} \tilde{x}_k + \tilde{B}_{\theta_k} f_k, \\ y_k &= \tilde{C}_{\theta_k} \tilde{x}_k, \ k \in \mathbb{Z}_+. \end{aligned}$$

The state vector $\tilde{x}_k$ is then partitioned as

$$\tilde{x}_k = \begin{bmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{bmatrix},$$

where the state vector $\tilde{x}_{1_k}$ corresponds to the states that are to be retained and $\tilde{x}_{2_k}$ to the states that are to be removed. With appropriate partitioning of the system matrices the state space representation of $\mathcal{G}$ becomes

$$\begin{aligned} \tilde{x}_{1_{k+1}} &= \tilde{A}_{11_{\theta_k}} \tilde{x}_{1_k} + \tilde{A}_{12_{\theta_k}} \tilde{x}_{2_k} + \tilde{B}_{1_{\theta_k}} f_k, \\ \tilde{x}_{2_{k+1}} &= \tilde{A}_{21_{\theta_k}} \tilde{x}_{1_k} + \tilde{A}_{22_{\theta_k}} \tilde{x}_{2_k} + \tilde{B}_{2_{\theta_k}} f_k, \\ y_k &= \tilde{C}_{1_{\theta_k}} \tilde{x}_{1_k} + \tilde{C}_{2_{\theta_k}} \tilde{x}_{2_k}, \qquad k \in \mathbb{Z}_+. \end{aligned}$$

The dynamic system that one obtains by truncating the last r variables, i.e. $\tilde{x}_{2_k} \in \mathbb{R}^r$, is equivalent to a system whose state variables are constrained in a proper subspace S_{n-r} of the original state space, where $S_{n-r} = \{x \in \mathbb{R}^n \mid x(i) = 0, \ \ n-r+1 \leq i \leq n\}$, that is naturally isomorphic to $\mathbb{R}^{n-r}$. Thus the state vector $\hat{x}_k$ of the reduced system $\hat{\mathcal{G}}$ will be of the form $\hat{x}_k = (\tilde{x}_{1_k}, 0)' \in S_{n-r} \subset \mathbb{R}^n$.

III. Upper bound to the approximation error

In this section it will be shown how to reduce the order of a given mean square stable system $\mathcal{G}$ by means of state truncation and obtain an upper bound on the stochastic L_2 gain of the resulting error system $\mathcal{E}$.

Theorem 3.1: Consider a mean square stable system $\mathcal{G}$ of order n. Consider also the positive definite matrix W, such that

$$W = \Sigma_1 \oplus \Sigma_2,$$

where

$$\Sigma_2 = \beta_1 I_{r_1} \oplus \ldots \oplus \beta_s I_{r_s}, \qquad \sum_{k=1}^{s} r_k = r.$$

Suppose that the matrix $P = W$ satisfies (7) and $\hat{Q} = W^{-1}$ satisfies (8). Let $\tilde{\mathcal{G}}$ be the reduced order model obtained by truncating the last r states of $\mathcal{G}$. Then, the stochastic L_2 gain of the error system $\mathcal{E}$ is bounded from above by twice the sum of the distinct entries on the diagonal of Σ_2 :

$$\gamma_{\mathcal{E}} \leq 2(\beta_1 + \ldots + \beta_s) \tag{10}$$

Proof. Using the matrix

$$E_r = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}$$

the state space representation of $\hat{\mathcal{G}}$ can be written as

$$\begin{aligned} \hat{x}_{k+1} &= (I_n - E_r)(A_{\theta_k} \hat{x}_k + B_{\theta_k} f_k), \\ \hat{y}_k &= C_{\theta_k} \hat{x}_k, \qquad k \in \mathbb{Z}_+. \end{aligned} \tag{11}$$

The following signals will shorten the subsequent notation.

$$\begin{aligned} z_k &= x_k + \hat{x}_k, \\ \delta_k &= x_k - \hat{x}_k \\ h_{\theta_k} &= A_{\theta_k} \hat{x}_k + B_{\theta_k} f_k, \quad \theta_k \in \Theta. \end{aligned}$$

The proof will proceed by successive truncation of the last $r_s, r_{s-1}, \ldots, r_1$ states. Let $\mathcal{G}_s$ denote the reduced system obtained by truncating the last r_s states and $\mathcal{E}_s$ the corresponding error system between $\mathcal{G}_s$ and $\mathcal{G}$. The state variable of $\mathcal{G}_s$ is $\hat{x}^{(s)} \in S_{n-r_s} \subset \mathbb{R}^n$ and one can verify that the following relations hold:

$$\begin{aligned} z_{k+1}^{(s)} &= A_{\theta_k} z_k^{(s)} + 2B_{\theta_k} f_k - E_{r_s} h_{\theta_k}^{(s)}, \\ \delta_{k+1}^{(s)} &= A_{\theta_k} \delta_k^{(s)} + E_{r_s} h_{\theta_k}^{(s)}, \\ e_k^{(s)} &= C_{\theta_k} \delta_k^{(s)}, \qquad k \in \mathbb{Z}_+, \end{aligned}$$

where

$$\begin{aligned} z_k^{(s)} &= x_k + \hat{x}_k^{(s)}, \\ \delta_k^{(s)} &= x_k - \hat{x}_k^{(s)}, \\ e_k^{(s)} &= y_k - y_k^{(s)}. \end{aligned}$$

In a first step it will be shown that

$$\gamma_{\mathcal{E}_s} \leq 2\beta_s \tag{12}$$

In order to prove (12) one can follow arguments similar to Lemma 1.1. Namely, it is sufficient to find a storage function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, such that $V(0,0) = 0$ and :

$$\begin{aligned} &\Psi(x, \hat{x}^{(s)}, f) \geq 0, \\ &\forall x \in \mathbb{R}^n,\ \forall \hat{x}^{(s)} \in S_{n-r_s},\ \forall f \in \mathbb{R}^m, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Psi(x, \hat{x}^{(s)}, f) &= 4\beta_s^2 |f|^2 - \sum_{i=1}^N q_i |C_i\ \delta^{(s)}|^2 - \Delta V, \\ \delta^{(s)} &= x - \hat{x}^{(s)} \\ \Delta V &= \sum_{i=1}^N q_i V(x_+, \hat{x}_+^{(s)}) - V(x, \hat{x}^{(s)}) \\ x_+ &= A_i x + B_i f \\ \hat{x}_+^{(s)} &= (I_n - E_{r_s})(A_i \hat{x}^{(s)} + B_i f) \end{aligned}$$

Note that the above set of relations essentially imply

$$\begin{aligned} 0 \leq\ & 4\beta_s^2 |f_k|^2 + \\ -\ & \mathbf{E}[|E_k^{(s)}|^2 + V(X_{k+1}, \hat{X}_{k+1}^{(s)}) - V(X_k, \hat{X}_k^{(s)})], \\ & \forall f_k \in \mathbb{R}^m \end{aligned}$$

and thus (12). A quadratic storage function candidate is given by :

$$V(x, \hat{x}^{(s)}) = \beta_s^2 |z^{(s)}|^2_{W^{-1}} + |\delta^{(s)}|^2_W$$

In order to verify (13) one needs to compute the expected increment of the storage function along system trajectories.

$$\begin{aligned} \Delta V = & \sum_{i=1}^N q_i |A_i \delta^{(s)} + E_{r_s} h_i^{(s)}|^2_W + \\ & \beta_s^2 \sum_{i=1}^N q_i |A_i z^{(s)} + 2B_i f - E_{r_s} h_i^{(s)}|^2_{W^{-1}} + \\ & -\beta_s^2 |z^{(s)}|^2_{W^{-1}} - |\delta^{(s)}|^2_W. \end{aligned}$$

Expanding the individual term in the above expressions, one obtains

$$\begin{aligned} \Delta V = & \sum_{i=1}^N q_i |A_i \delta^{(s)}|^2_W - |\delta^{(s)}|^2_W + \\ & +\beta_s^2 \sum_{i=1}^N q_i |A_i z^{(s)} + 2B_i f|^2_{W^{-1}} - \beta_s^2 |z^{(s)}|^2_{W^{-1}} \\ & +2\beta_s \sum_{i=1}^N q_i |E_{r_s} h_i^{(s)}|^2 \\ & -2\beta_s \sum_{i=1}^N q_i (E_{r_s} h_i^{(s)})'(A_i z^{(s)} + 2B_i f - A_i \delta^{(s)}). \end{aligned} \tag{14}$$

Applying the dissipation inequality (7) on the first two terms of (14) gives

$$\sum_{i=1}^N q_i |A_i \delta^{(s)}|^2_W - |\delta^{(s)}|^2_W \leq -\sum_{i=1}^N q_i |C_i \delta^{(s)}|^2.$$

Using the dissipation inequality (8), the second line in (14) becomes

$$\beta_s^2 \sum_{i=1}^N q_i |A_i z^{(s)} + 2B_i f|^2_{W^{-1}} - \beta_s^2 |z^{(s)}|^2_{W^{-1}} \leq 4\beta_s^2 |f|^2.$$

For the last term of (14) note that

$$A_i z^{(s)} + 2B_i f - A_i \delta^{(s)} = 2h_i^{(s)},$$

and that $E_{r_s}^2 = E_{r_s}$. Using the above relations we obtain

$$\begin{aligned} \Delta V \leq\ & -\sum_{i=1}^N q_i |C_i \delta^{(s)}|^2 + 4\beta_s^2 |f|^2 - \\ & 2\beta_s \sum_{i=1}^N q_i |E_{r_s} h_i^{(s)}|^2. \end{aligned}$$

Substitute the above inequality in (14) to obtain

$$\begin{aligned} &\Psi_k(x, \hat{x}^{(s)}, f) \geq 2\beta_s \sum_{i=1}^N q_i |E_{r_s} h_i^{(s)}|^2 \geq 0, \\ &\forall \hat{x}^{(s)} \in S_{n-r_s}, \forall f \in \mathbb{R}^m, \end{aligned}$$

completing the first part of the proof. Let W_s be a submatrix of W corresponding to the retained states.

$$W_s = \Sigma_1 \oplus \beta_1 I_{r_1} \oplus \ldots \oplus \beta_s I_{r_{s-1}}.$$

Note that W_s satisfies the generalized dissipation inequalities corresponding to $\mathcal{G}_s$, in the sense

$$\begin{aligned} \sum_{i=1}^N q_i (|A_i \hat{x}^{(s)}|^2_{W_s} + |C_i \hat{x}^{(s)}|^2) &\leq |\hat{x}^{(s)}|^2_{W_s}, \\ & \forall \hat{x}^{(s)} \in S_{n-r_s}, \\ \sum_{i=1}^N q_i (|A_i \hat{x}^{(s)} + B_i f|^2_{W_s^{-1}}) &\leq |\hat{x}^{(s)}|^2_{W_s^{-1}} + |f|^2, \\ & \forall \hat{x}^{(s)} \in S_{n-r_s}, \forall f \in \mathbb{R}^m. \end{aligned}$$

Thus, if the last r_{s-1} states from $\mathcal{G}_s$ are truncated and if one denotes the resulting system $\mathcal{G}_{s-1}$ and the corresponding error system between $\mathcal{G}_s$, $\mathcal{G}_{s-1}$ by $\mathcal{E}_{s-1}$ then by repeating the above argument

$$\gamma_{\mathcal{E}_{s-1}} \leq 2\beta_{s-1}$$

Similarly,

$$\gamma_{\mathcal{E}_j} \leq 2\beta_j \quad j \in \{s, s-1, \ldots, 1\}.$$

The desired result (10) is obtained by observing that $e_k = e_k^{(1)} + \ldots + e_k^{(s)}$ and applying the triangle inequality on stochastic L_2 gains. ■

A. Obtaining $W = \Sigma_1 \oplus \Sigma_2$, W diagonal

The theorem of the previous section assumes there exists a $W = \Sigma_1 \oplus \Sigma_2$, Σ_2 diagonal, such that $W = P$ satisfies (7) and $\hat{Q} = W^{-1}$ satisfies (8). In this section it will be shown that under the standing assumption of mean square stability, one can obtain in fact a diagonal matrix W with the desired properties.

Mean square stability is equivalent with the existence of $\hat{P} > 0$, such that

$$\sum_{i=1}^{N} q_i A_i' \hat{P} A_i - \hat{P} < 0. \qquad (15)$$

Relation (7) is equivalent to

$$\sum_{i=1}^{N} q_i A_i' P A_i - P \leq -\sum_{i=1}^{N} q_i C_i' C_i. \qquad (16)$$

By virtue of the above two relations, if one sets $P = \alpha \hat{P}$ and takes $\alpha > 0$ large enough, the dissipation inequality (7) can always be satisfied by some positive definite matrix P. Relation (8) is equivalent to

$$\begin{bmatrix} -\hat{Q} + \sum_{i=1}^{N} q_i A_i' \hat{Q} A_i & \sum_{i=1}^{N} q_i A_i' \hat{Q} B_i \\ \sum_{i=1}^{N} q_i B_i' \hat{Q} A_i & -I + \sum_{i=1}^{N} B_i' \hat{Q} B_i \end{bmatrix} \leq 0 \qquad (17)$$

Note that, if one sets $\gamma = 1$, $C_i = 0, \forall i \in \{1, \ldots, N\}$, and $\hat{Q} = \alpha P$ with $\alpha > 0$, the above relation becomes equivalent to (4). Let $\hat{P}$ satisfy (15) and set $\hat{Q} = \alpha \hat{P}$ with $\alpha > 0$. Condition (5) is equivalent to mean square stability and thus feasible for all positive values of α. Condition (6) can be rewritten as

$$\sum_{i=1}^{N} q_i B_i' \hat{Q} B_i - W_{21} W_{11}^{-1} W_{12} < I.$$

Both terms on the left hand side of the above relation scale linearly with α. Thus, by taking the positive parameter α to be small enough one can also satisfy (8) with the choice of $\hat{Q} = \alpha \hat{P}$. To this point one has obtained $P > 0$ and $\hat{Q} > 0$ such that (7) and (8) are feasible. What remains then, is to compute a transformation matrix T that diagonalizes the product $P\hat{Q}^{-1}$. In that case $TP\hat{Q}^{-1}T^{-1} = W^2 > 0$ and (7) is satisfied by W and (8) by W^{-1}, justifying the assumption of the previous theorem in regards to W.

B. Non uniqueness of P and $\hat{Q}$

In general LMI's may have multiple solutions, and thus there is no unique solution to (7) and (8). However the dissipation inequality (7) and its equivalent form (16) possess a unique minimal solution, which can be computed by solving the linear algebraic equation :

$$\sum_{i=1}^{N} q_i (A_i' P A_i + C_i' C_i) = P$$

When it comes to relation (8) or its equivalent form (17) the situation is different. For $N = 1$, the inverse of the controllability grammian corresponds to a maximal solution of (17). In the case of an LPV system, where $N > 1$, there is no maximal solution though. For example, let N = 2 and

$$\hat{Q}_i = \arg\max_{\hat{Q}>0} \operatorname{trace}(R_i \hat{Q}), \quad i \in \{1, 2\} \qquad (18)$$

subject to (17), where $R_i \geq 0, i \in \{1, 2\}$. If there was a maximal solution to (17), then one should have

$$\hat{Q}_1 = \hat{Q}_2 \qquad (19)$$

The following system shows that (19) is not satisfied. Let

$$\begin{aligned} R_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.0 \\ 0.3 & 0.5 \end{bmatrix}, \\ R_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}, \\ B_1 &= B_2 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}, \quad q_1 = q_2 = \frac{1}{2}. \end{aligned}$$

Solving the optimization problem (18) subject to (17) gives

$$\hat{Q}_1 = \begin{bmatrix} 17.4 & -16.7 \\ -16.7 & 21.3 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} 14.9 & -16.0 \\ -16.0 & 24.9 \end{bmatrix}.$$

The lack of a maximal solution to (17), is to some extent unfortunate, since the diagonal entries of Σ_2 that appear in (10) are monotonic in P and $\hat{Q}^{-1}$. A reasonable remedy is to compute a positive definite matrix $\hat{Q}$ such that $\operatorname{trace}(P^{-1}\hat{Q})$ is maximized subject to the constraint (17). The motivation for this objective function comes from the fact that

$$\operatorname{trace}(P^{-1}\hat{Q}) = \sum_{i=1}^{N} \frac{1}{\beta_i^2}$$

and thus the smaller eigenvalues of W are more heavily penalized in this optimization criterion, which is desirable given the nature of the error bound (10).

IV. A NUMERICAL EXAMPLE OF THE METHOD

The reduction method will be demonstrated on a simple example that involves a system $\mathcal{G}$ with 2 modes, 2 states, 1 input and 1 output having the system matrices :

$$A_1 = \begin{bmatrix} \beta + \alpha & 0 \\ 0 & \beta - \alpha \end{bmatrix}, \quad A_2 = \begin{bmatrix} \beta - \alpha & 0 \\ 0 & \beta + \alpha \end{bmatrix}$$

where β and α are positive parameters.

$$B_1 = B_2 = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1' = C_2' = C'.$$

Note that the above system is worst case stable if and only if $\beta + \alpha < 1$. The parametric input is randomized by setting $q_1 = q_2 = \frac{1}{2}$. System $\mathcal{G}$ is mean square stable if and only if $\beta^2 + \alpha^2 < 1$. As expected the requirement of stochastic stability relaxes the constraints on the parameters α, β. Let

$$W = \begin{bmatrix} \lambda_{max} & 0 \\ 0 & \lambda_{min} \end{bmatrix},$$

and set $\beta = 0.7$. The ratio $\frac{\lambda_{max}}{\lambda_{min}}$ is depicted in the following figure as a function of α. Given the nature of the error bound,

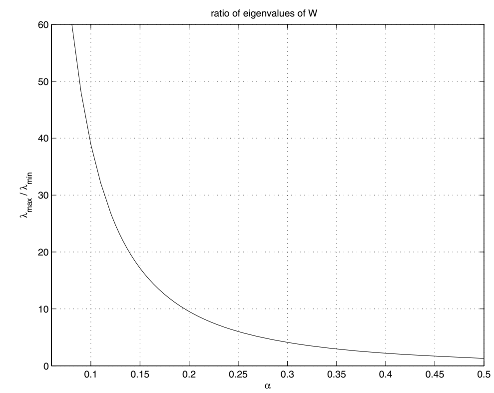

Fig. 2. Ratio $\frac{\lambda_{max}}{\lambda_{min}}$ of the eigenvalues of W, ($\beta = 0.7$, $\alpha \in [0.06, 0.5]$).

one can expect, that the larger the eigenvalue ratio of W the better the quality of the reduction. Note that as α converges to 0, $\mathcal{G}$ converges to a first order linear time-invariant system. Truncating one state from $\mathcal{G}$ leads to a reduced system $\hat{\mathcal{G}}$, that turns out to be a linear time invariant system with a single pole at β. The response of the two systems to a step input for a particular realization of the parametric input is depicted in the following figure for $\beta = 0.7$ and $\alpha = 0.1$.

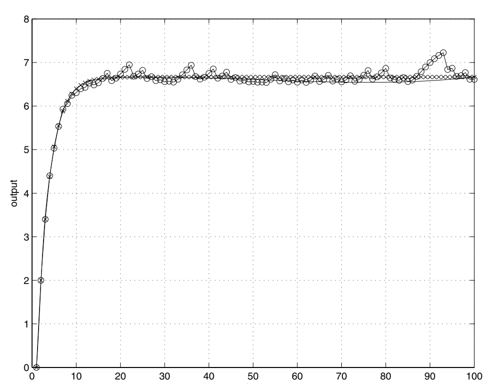

Fig. 3. Response of $\mathcal{G}$ and $\hat{\mathcal{G}}$ to a step input for a particular realization of the parametric input θ_k, ($\beta = 0.7$, $\alpha = 0.1$).

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