Necessary and Sufficient Conditions for Robust Stability of a Class of Nonlinear Systems

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Abstract

Input-output stability results for feedback systems are developed. Robust Stability conditions are presented for nonlinear systems with nonlinear uncertainty defined by some function (with argument equal to the norm of the input) that bounds its output norm. A sufficient small gain theorem for a class of these systems is presented. Then it is also shown that, for the vector spaces $\ell_2$ and $\ell_\infty$, the sufficient conditions are also necessary with some additional assumptions on the systems. These results capture the conservatism of the small gain theorem as it is applied to systems that do not need to have linear gain.

1 Introduction

This paper considers the development of necessary and sufficient conditions for the robust stabilization of certain classes of nonlinear plants. The problem of robust stabilization may be stated as follows. Given a nominal plant model and a family of possible true plants, under what conditions does some compensator which stabilizes the nominal plant also stabilize every plant in the given family?

The idea that a loop of less than unity gain ensures stability of a feedback loop has been appreciated since the early days of classical control. In mathematical terms, it is related to well-known ideas on invertibility of nonlinear operators of the form $\delta + G_2G_2$ where $\delta$ is the identity and $G_1, G_2$ are nonlinear operators on Banach spaces.

The usual form of the small gain theorem assumes gain properties of the form

$$\|M\|_\ell \leq \gamma$$

for the operator $M$ where $u$ denotes the input signal, $\gamma = \sup_{x \neq 0} \|Mx\|$, and $u_T$ denotes the truncation of the signal $u$ at time $T$. With this structure, it is shown that, if $M$ is linear, for $p = 2, \infty$ if $\|\Delta\|_{\ell_{p-n}} < 1$ then the feedback system of $\Delta$ and $M$ achieves robust stability if and only if $\|M\|_{\ell_{p-n}} \leq 1$.

Formal, see [1]. For a nonlinear $M$, the result is shown only for $p = 2$ (see [6] or [7]).

In [5] a new notion of stability for nonlinear systems is introduced. There, a generalization of (1) is given to allow more general bounding functions of the form

$$\|M\|_\ell \leq f(\|u_T\|)$$

where $f(\cdot)$ is a monotone function. Systems satisfying the last inequality are called monotone stable.

While conditions for sufficiency were shown with this new notion of nonlinear gain (also used by others like [8] or [9]) no results on the necessity of such conditions are known. Such results are useful to understand the degree of conservatism that the small gain theorem has. Necessity conditions for linear gain exist (see [1] or [3]). There also exists necessity conditions for nonlinear systems that have their output norm bounded by a linear function of the input norm (see [6] or [7]).

In this paper, an extension of the small gain theorem presented in [5] will be given. Using this theorem we will present sufficient conditions on $M$ in order to guarantee robust stability. For a certain class of perturbations, these conditions on $M$ will be simplified. Then, for the vector space $\ell_2$, $\ell_\infty$ we will give conditions on $M$ so that the sufficient conditions are also necessary with either NLTV or NLTI perturbations. For the vector field $\ell_2$, $\ell_\infty$ we will also give conditions on $M$ so that the sufficient conditions are necessary with non causal perturbations. The construction of a causal perturbation is still under investigation.

2 Preliminaries

We start by defining some standard concepts. The set of nonnegative integers is denoted by $\mathbb{Z}_+$. The extended space of sequences in $\mathbb{R}^n$ is denoted by $\ell_p$ for every $1 \leq p \leq \infty$ or just by $\ell$ when it is obvious or when it just does not matter what $p - norm$ is being used. The restriction of $f$ to the interval $[a, b]$ is denoted by $f_{[a, b]}$. For every $f = \{f(0), f(1), \ldots\} \in \ell$

$\|f\|_p$ as $\|f\|_p = \left(\sum_{n=a}^{b} |f(n)|^p\right)^{1/p}$

The set of all $f \in \ell$ with $f \notin \ell_p$ is denoted by $\ell \setminus \ell_p$. Given $f \in \ell$ define $f^\sharp = (f(0)^*, f(1)^*, f(2)^*, \ldots)$ and the support of $f \in \ell$ by $\supp(f) = \{n : f(n) \neq 0\}$.

For $k \in \mathbb{Z}_+$, $S_k$ denotes the $k$th-shift (delay-time) operator $\ell$ and $P_k$ the $k$-th truncation operator on $\ell$. Let $H : \ell \to \ell$ be an operator. Then, $H$ is called causal if $P_kHf = P_kH_Pf, \forall k \in \mathbb{Z}_+$, strictly causal if $P_kHf = P_kP_{k-1}f, \forall k \in \mathbb{Z}_+$, and time invariant if $H_S = S_kH$.

Let $X_\ell$ and $Y_\ell$ be two signal spaces. Then an operator $G : X_\ell \to Y_\ell$ provides an input-output system representation.

The following definition provides a concept of input output stability.
Definition 2.1 The system $G$ is monotone stable if there exists a monotone increasing homeomorphism \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) and a constant $\beta \in \mathbb{R}_+$ such that
\[
\|(Gu)_T\| \leq f(\|u_T\|) + \beta
\]
for all $u \in X_+$ and $T \geq 0$.

Definition 2.2 A nonlinear operator $G$ is said to have finite memory if there exists an integer function $FM(\cdot; G) : Z_+ \to Z_+$ with $FM(t; G) \geq t$ such that
\[
(I - P_{FM(\cdot; G)})Gf = (I - P_{FM(\cdot; G)})G(I - P_t)f
\]
for all $f \in \ell_p$ and $t \in Z_+$. The function $FM(\cdot; G)$ is called the finite memory function associated with $G$.

The proof of the following proposition is done in [6] and therefore it is omitted here.

Proposition 2.1 Let $G$, a nonlinear operator, have finite-memory with associated finite-memory function $FM(\cdot; G)$. Then for $f_1 \in \ell_2$ with $\text{supp}(f_1) \subset [0, n]$ and $f_2 \in \ell_2$ with $\text{supp}(f_2) \subset [FM(n; G) + 1, \infty]$ we have $G(f_1 + f_2) = Gf_1 + Gf_2$.

In the following definition, assume that $G$ is some nonlinear operator.

Definition 2.3 Let $\eta_G(s) : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that
\[
\sup_{f \neq 0} \frac{\|G(f)\|}{\eta_G(\|f\|)} = 1
\]
If $G$ is finite memory, then there always exist an $f$ that achieves the norm. Therefore, the last definition can be understood as follows: there exists an $f \in \ell_p$ such that $\|G(f)\| = \eta_G(\|f\|)$; for all other $u \in \ell_p$, $\|G(u)\| \leq \eta_G(\|u\|)$.

Consider the system in figure 1.

Figure 1: Closed loop system

Let $\Delta$ denote the class of allowable perturbations. We now define the subset of $\Delta$ containing elements with $\eta_\Delta(s) < ks^2$, for some given $x$, $k > 0$ (if $k = 0$ then it is obvious that the system in figure 1 is stable if and only if $M$ is stable).

Definition 2.4 Let $C_{\Delta, p, x}$ be a subset of $\Delta$ defined, for some given $k > 0$, as
\[
C_{\Delta, p, x} = \{ \Delta \in \Delta : \eta_\Delta(s) < ks^2 \}
\]

This is the same to say that, for every $u \in \ell_p$ and $T \in Z_+$, $\|\Delta(uT)\|_p \leq \eta_\Delta(\|uT\|_p)$, where $\eta_\Delta(s) < ks^2$. This means that $\|\Delta(uT)\|_p < \|u\|_p^2$ and therefore, according to definition 2.1, all $\Delta \in C_{\Delta, p, x}$ are monotone stable (note that $ks^2$ is a monotonic increasing homeomorphism for every $x$, $k > 0$).

For perturbations $\Delta \in C_{\Delta, p, x}$ (for some $p$ and $x$), the problem will be to find necessary and sufficient conditions on $M$ to guarantee robust stability.

3 Sufficiency of the Small-Gain Theorem

In this section we will present a sufficient condition to achieve robust stability when the disturbance belongs to $C_{\Delta, p, x}$. First, in section 3.1, we will give some concepts that will be used in section 3.2 to present the general small-gain theorem. This theorem is an extension to the one presented in [5]. Finally, in section 3.3, using this theorem, a sufficient condition is given on some system $M$, perturbed by $\Delta \in C_{\Delta, p, x}$, that guarantees the robust stability of the feedback system in figure 1.

3.1 Preliminaries

Consider the system of figure 2.

Figure 2: Closed loop system

Assumption 3.1 Let $V_{1e}$ and $V_{2e}$ be two signal spaces. The operators $G_1 : V_{1e} \to V_{2e}$ and $G_2 : V_{2e} \to V_{1e}$ are such that for all input signals $r_1 \in V_{1e}$ and $r_2 \in V_{2e}$ there exist unique signals $u_1, u_2 \in V_{1e}$ and $u_2, y_1 \in V_{2e}$.

Definition 3.1 Define the following function classes $M = \{ f : \mathbb{R}_+ \to \mathbb{R}_+ \}$, $N = \{ f \in M \mid \exists y \in M \text{ s.t. } f(x) \leq x - g(x) \}$, and $N_0 = \{ f \in M \mid \exists y \in M \text{ s.t. } f(x) \leq x - g(x) \forall x \geq y \}$, where $y \geq 0$ and $\exists y \in N_0$ such that $\eta_\Delta(s) < ks^2$.

So, $N \subset N_0$. Define also $M0 = M \cup \{0\}$ and $N0 \cup \{0\}$ where $0$ denotes the zero function $f \equiv 0$.

Proposition 3.1 $f \in N_0$ if and only if $\exists y \in M$ such that $(i + g) \circ f(x) \leq x$ for all $x \geq y$.

Definition 3.2 The feedback system in figure 2 under assumption 3.1 is called monotone stable if there
exist functions \( f_1, f_2 : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and constants \( \beta_1, \beta_2 \in \mathbb{R} \) such that
\[
\|y_t\| \leq f_1(\|r_{1t}\|, \|r_{2t}\|) + \beta_1
\]
\[
\|y_t\| \leq f_2(\|r_{1t}\|, \|r_{2t}\|) + \beta_2
\]
\( \forall T \geq 0, \forall r_t \in \ell_{p_1}, \forall r_t \in \ell_{p_2}, \) and \( f_1(0,0), f_2(0,0), f_1(\cdot, f_2(\cdot, 0)) \in M_0 \) with \( f_1(0,0) = f_2(0,0) = 0 \).

### 3.2 General case

Let each system be monotone stable with gain functions \( g_1 \) and \( g_2 \) as in definition 2.1. This means that
\[
\|y_t\| \leq g_1(\|u_{1t}\|) + \beta_{g_1}
\]
\[
\|y_t\| \leq g_2(\|u_{2t}\|) + \beta_{g_2}
\]

The proof of the following theorem is similar to the one in [5] and therefore it will be omitted here.

**Theorem 3.1** Consider the system in figure 2. Suppose \( G_1 \) and \( G_2 \) are stable with gain functions \( g_1 \) and \( g_2 \) as in (3.4). Suppose that assumption 3.1 holds. The feedback system is monotone stable if there exist \( g \in M \) and \( s^* \geq 0 \) such that
\[
g_2 \circ (i + g) \circ g_1 \in N_{s^*}.
\]

**Comment:** In the proof of the last theorem (not done here), \( s^* \neq 0 \) implies that \( \beta_1 = 0 \) and \( \beta_2 = 0 \) in definition 3.2 although both systems \( G_1 \) and \( G_2 \) have zero bias terms. The reason for is that the \( \beta_i \) are used to accommodate the lack of information on the closed loop system for \( s \leq s^* \) (in equation (5)). Since (5) only gives us information for \( s \leq s^* \), we need \( \beta_1 
eq \beta_2 
eq 0 \) in order to bound \( \|y_{1t}\| \) and \( \|y_{2t}\| \). Note that if \( s^* = 0 \) we have exactly the theorem presented in [5].

**Corollary 3.1** Consider the system in figure 2. Suppose \( G_1 \) and \( G_2 \) are stable with gain functions \( g_1 \) and \( g_2 \) as in (3.4). Suppose that assumption 3.1 holds. The feedback system is monotone stable if there exist \( \rho_1, \rho_2 \in M \) and \( s^* \geq 0 \) such that
\[
(i + \rho_1) \circ g_2 \circ (i + \rho_2) \circ g_1 \leq s \quad \text{for all} \ s \geq s^*
\]

### 3.3 Particular case

Consider again the system in figure 1. Assume that \( \Delta \in C_{\Delta_{p,\infty}} \) and that \( M \) is monotone stable with gain function \( m(s) \). For simplicity, let \( \delta(s) = \eta_\Delta(s) \).

We will prove that it is sufficient to have \( m(s) \leq \left( \frac{\epsilon}{\Delta} \right)^{\frac{1}{2}} \) (and therefore \( \delta \circ m)(s) \leq \epsilon \) for all \( s \geq s^* \) for some \( s^* \geq 0 \) in order to have robust stability. This means that we do not need to find monotonically increasing functions \( \rho_1 \) and \( \rho_2 \) satisfying \( (i + \rho_1) \circ \delta \circ (i + \rho_2) \circ m(s) \leq s \) in order to have closed loop stability.

**Theorem 3.2** The system in figure 1 achieves robust stability for all \( \Delta \in C_{\Delta_{p,\infty}} \) if there exists \( s^* \geq 0 \) such that
\[
m(s) \leq \left( \frac{\epsilon}{\Delta} \right)^{\frac{1}{2}} \quad \text{for all} \ s \geq s^*.
\]

**Proof:** Assume there exists an \( s^* \geq 0 \) such that
\[
m(s) \leq \left( \frac{\epsilon}{\Delta} \right)^{\frac{1}{2}} \quad \text{for all} \ s \geq s^*.
\]

Also, because \( \Delta \in C_{\Delta_{p,\infty}} \), we have that \( \delta(s) < k s^* \). Then, for a given \( \Delta \in C_{\Delta_{p,\infty}} \), we can always find \( k_1 > 0 \) with \( k_1 < k \) such that \( \delta(s) < k_1 s^* \) for all \( s \geq 0 \). This means that
\[
\|y_{1t}\| < k_1 \|u_{1t}\| \quad \text{and therefore} \ \Delta \text{ is monotone stable with gain function} \ f(s) = k_1 s^*
\]

So, if we find \( \rho_1, \rho_2 \in M \) and \( s \geq 0 \) such that (7)
\[
(i + \rho_1) \circ m \circ (i + \rho_2) \circ f(s) \leq s
\]

we meet all the conditions of corollary 3.1 and therefore we prove stability.

Let \( \rho_1(s) = \beta s \) and \( \rho_2(s) = \delta s \) (with \( \beta > 0 \) and \( \delta > 0 \)). Then
\[
(i + \rho_1) \circ \delta \circ (i + \rho_2) \circ f(s) = m(k_1 s^*(1 + \epsilon))(1 + \beta)
\]

Let \( \delta = \left( \frac{\epsilon}{k_1(1 + \epsilon)} \right)^{\frac{1}{2}} \). Note that when \( s^* = 0 \), we have \( \delta = 0 \). Then, for any \( s \geq \delta, m(k_1 s^*(1 + \epsilon)) \leq \left( \frac{\epsilon}{k_1} \right)^{\frac{1}{2}} k_1 s^*(1 + \epsilon) \). Therefore, from (8)
\[
(i + \rho_1) \circ m \circ (i + \rho_2) \circ f(s) \leq s \left( \frac{k_1}{k} \right)^{\frac{1}{2}} s \left( 1 + \epsilon \right)(1 + \beta)
\]

and if we let \( \epsilon = \frac{1 - \sqrt{\frac{\beta}{2}}}{2\sqrt{\frac{\beta}{2}}} > 0 \) and \( \beta = \frac{1 - \sqrt{\frac{\beta}{2}}}{2\sqrt{\frac{\beta}{2}}} > 0 \)

we actually satisfy (7) for all \( s \geq \delta \) which implies (from corollary 3.1) that the feedback system is stable.

### 4 \( \ell_{\infty} \) Stability Robustness Necessary Conditions

Consider the system in figure 1. In [1, 2, 4] necessary conditions for stability robustness were presented for the case when \( M \) is linear time invariant. We will now extend those conditions to certain classes of nonlinear \( M \). First, we will consider the case where the perturbation is NTV. Then, we will prove that the necessity conditions still hold if the perturbation is NLTI.

To prove necessity, we add the following assumption on \( M \).

**Assumption 4.1** Assume that the bound \( \eta_M \) defined in definition 2.3 satisfies, for all \( s \geq 0 \), \( \eta_M(s) = \sup_{\|f\|_{\infty} = s} \|M(f)\|_{\infty} \).

#### 4.1 \( \ell_{\infty} \) Stability robustness with NTV perturbations

Assume that \( C_{\Delta_{TV,\infty}} \) represents the set of all causal NTV perturbations according to definition 2.4. Also, \( M \) is stable, causal, and NLTV.

**Theorem 4.1** Under assumption 4.1, the system in figure 1 achieves robust stability for all \( \Delta \in C_{\Delta_{TV,\infty}} \) if and only if there exists an \( s^* \geq 0 \) such that \( \eta_M(s) \leq \left( \frac{\epsilon}{\Delta} \right)^{\frac{1}{2}} \) for all \( s \geq s^* \).
Proof: We first prove sufficiency. Assume that there exists an \( s^* \geq 0 \) such that \( \eta_M(s) \leq (\ell)^{2} \) for all \( s \geq s^* \). This is the same to say that for any \( u \in \ell \) with \( \| u \| \geq s^* \), \( \| M(u) \| \leq (\ell)^{2} \) which means that \( M \) is monotone stable with gain function \( m(s) = (\ell)^{2} \) for every \( s \geq s^* \). We can now use theorem 3.2 and conclude that the closed loop system is stable.

We now prove necessity. To simplify the proof, consider \( M \) and \( \Delta \) SISO and let \( m(s) = \eta_M(s) \) and \( \delta(s) = \eta_\Delta(s) \).

The approach we use is to show that we can construct a destabilizing perturbation \( \Delta \in C_{\Delta_{TV, \infty, z}} \) whenever the conditions of the theorem are not satisfied. So, assume that \( \forall s \geq 0, \exists s > s^* : m(s) > (\ell)^{2} \).

As in [1, 2, 4], the proof is divided in two parts: construction of an unbounded signal and construction of a destabilizing perturbation using that signal.

Construction of the unbounded signals

For simplicity assume that \( M \) has finite-memory. This means that there exists an increasing integer function \( F_M(\cdot; M) \) as in definition 2.2.

![Figure 3: Construction of \( \xi \)]

Assume that \( N_0 = 0 \) and \( s_0 = 0 \). The construction of \( \xi \) proceeds as follows (see figure 3). For all \( n = 1, 2, 3, \ldots \), let \( N_n = F_M(N_{n-1}; M) \) and \( s_n = k \left( \frac{s_{n-1}}{k} \right)^2 + 1 \). Then \( \exists s_n \geq s_n^* : m(s_n) > (\ell)^{2} \).

Choose \( \| \xi(t) \| \leq s_n \) for \( t = N_{n-1}, \ldots, N_n - 1 \) with \( \| P_{N_{n-1}} \| \geq s_n \) such that \( m(s) \) is achieved. Then \( \| P_{N_{n-1}} \| > (\ell)^{2} \) and \( \| P_{N_{n-1}} \| > (\ell)^{2} + 1 \).

Therefore \( \| P_{N_{n-2}} \| > \left( \frac{1}{k} \right)^{2} \) and \( \| P_{N_{n-1}} \| > (\ell)^{2} + 1 \).

Note that, in this case, \( \| P_{N_{n-1}} \| \geq s_n \geq k \left( \frac{s_{n-1}}{k} \right)^2 + 1 \) and \( \| P_{N_{n-1}} \| > (\ell)^{2} + 1 \).

Because of the way \( \xi(t) \) was constructed, we have

\[
\| P_{N_{n-1}} \| > \left( \frac{1}{k} \right)^{2} \]  

(9)

for all \( t \).

Construction of the destabilizing perturbation

We have \( \xi = \{ \xi(t) \}_{t \geq 0} \in \ell \) and \( y = \{ y(t) \}_{t \geq 0} \in \ell \) such that (9) is satisfied. Note that the inequality in (9) is equivalent to \( \| P_{N_{n-1}} \| < k \| P_{N_{n-1}} \| \).

Constructing the destabilizing perturbation the same way as in [1, 2, 4] we have that \( \Delta \) is trivial if \( y = 0 \); just pick \( \Delta \) itself to be zero. So, assume that \( y \neq 0 \). Constructing \( \{ y(t_1), y(t_2), \ldots \} \) as in [1, 2, 4] we can now construct our \( \Delta \).

So, \( \Delta \) is constructed by having (see figure 4) \( \xi = \Delta(y) = \Delta y^2 \). This can be seen as a series of two systems. The first raises every element of \( y(t) \) to the power of \( x \) (and it is therefore nonlinear) while the second (\( \Delta \)) is just an LTV system.

![Figure 4: Structure of \( \Delta \)]

\( \Delta \) is a matrix constructed as in [1, 2, 4].

Each row of \( \Delta \) has at most one nonzero element which has absolute value less than \( k \). This means that \( \| \Delta \|_{\ell^\infty} < k \).

Now, let's see if \( \Delta \) belongs to the set \( C_{\Delta_{TV, \infty, z}} \). For every \( t \geq 0 \) we have \( \| P_{t} \xi \| = \| \Delta P_{t} y^2 \| < k \| P_{t} y^2 \| < k \| P_{t} y \| \) or just that \( \| P_{t} \xi \| < k \| P_{t} y \| \) which means that \( \Delta \) is in \( C_{\Delta_{TV, \infty, z}} \). Moreover, \( \Delta \) is causal and NLT.

So, we found a bounded input that produces an unbounded output. This means that in definition 3.2 there is no monotonic increasing homeomorphism \( f_1 \) such that \( \| y(t) \| \leq f_1 \| y(t) \| + \| \xi(t) \| + \| \delta(t) \| \) because there exists a bounded \( r_2 \) (with \( r_2 = 0 \)) that produces an unbounded \( y_1 \). Therefore, we conclude that the closed loop system is unstable.

Remark 4.1 For \( x = 1 \) the above theorem provides a necessity proof for \( \ell_{\infty} \) stability of finite memory systems that satisfies

\[
\eta_M(s) = \sup_{\| f \| = s} \| M(f) \| = s
\]

Moreover, the destabilizing perturbation can be LTV.

Remark 4.2 Assumption 4.1 which states that the supremum is achieved for each \( s \) can be relaxed. Instead, let \( s_k \) be a sequence with the properties

1. \( s_k - s_{k-1} \leq L \) for some \( L > 0 \) and
2. \( \lim_{k \to \infty} s_k = \infty \)

The above proof can be modified with this new assumption:

\[
\eta_M(s_k) = \sup_{\| f \| = s_k} \| M(f) \|
\]

The proof will be omitted.

4.2 \( \ell_{\infty} \) stability robustness with NLT perturbations

Assume here that \( C_{\Delta_{TV, \infty, z}} \) represents the set of all NLT perturbations according to definition 2.4. The proof of the following theorem is similar to the one done in [1].
Theorem 4.2 Under assumption 4.1, the system in figure 1 achieves robust stability for all \( \Delta \in C_{\Delta_{\mathbb{R}^{2}, \infty, \mathbb{x}}} \) and if only if there exists an \( s^* \geq 0 \) such that \( \eta_M(s) \leq (\delta) \frac{1}{k} \) for all \( s \geq s^* \).

Proof: The proof of this theorem follows exactly as the proof of theorem 4.1 except for the construction of the destabilizing perturbation. Given the signals \( y \) and \( \xi \), we show that a nonlinear time invariant perturbation can be constructed to destabilize the closed-loop system. Let the signals \( y \) and \( \xi \) be given as above. Then \( \Delta \) must be such that

\[
\delta(s) \leq \| \Delta \| \ell_{\infty, \infty} s^* < k s^*
\]

(10)

and \( \xi = \Delta \langle y \rangle \). We just need to redefine \( \Delta \). So, let \( \Delta \) be defined as follows

\[
(\Delta f)(t) = \begin{cases} k \xi(t - j), & \text{if } \exists j \in \mathbb{Z}_+ : P_1 f = P_2 S_j g, \\ 0, & \text{otherwise}. \end{cases}
\]

It is easy to see that the new \( \Delta \) is a nonlinear, time invariant, and causal system. It satisfies (10) (because \( \| \Delta \| \ell_{\infty, \infty} < k \)) which means that \( \Delta \in C_{\Delta_{\mathbb{R}^{2}, \infty, \mathbb{x}}} \) and maps \( y \) to \( \xi \).

5 \( \ell_2 \) Stability Robustness Necessary Conditions

Once again, we will extend the conditions for stability robustness presented in [1] to certain classes of nonlinear \( M \).

To prove necessity, we add the following assumption on \( M \), similar to the one in the \( \ell_\infty \) case.

Assumption 5.1 Assume that the bound \( \eta_M \) defined in definition 2.3 satisfies, for all \( s \geq 0 \), \( \eta_M(s) = \sup_{\| f \| = 1} \| M(f) \|_2 \).

5.1 \( \ell_2 \) stability robustness with non-causal perturbations

The following theorem gives a necessary and sufficient condition on the system \( M \) in figure 1 in order to guarantee that the closed-loop system is stable. Here, \( M \) is assumed to be some NLT system with its output norm bounded (to an input \( u \)) by \( \eta_M(\|u\|_2) \) according to definition 2.3 and assumption 5.1.

Assume here that \( C_{\Delta_{\mathbb{N}, \mathbb{R}^{2}, \mathbb{x}}} \) represents the set of all non causal perturbations according to definition 2.4.

Theorem 5.1 Let \( x \leq 1 \). Under assumption 5.1, the system in figure 1 achieves robust stability for all \( \Delta \in C_{\Delta_{\mathbb{N}, \mathbb{R}^{2}, \mathbb{x}}} \) if and only if there exists an \( s^* \geq 0 \) such that \( \eta_M(s) \leq (\delta) \frac{1}{k} \) for all \( s \geq s^* \).

Proof: The proof of sufficiency follows the same way as in the proof of the sufficiency of theorem 4.1. The method of proof for necessity will again be similar to the one in [1] or in [6, 7]. Once again, for simplicity, let \( m(s) = \eta_M(s) \) and \( \delta(s) = \eta_\Delta(s) \).

We will show that one can construct a destabilizing perturbation \( \Delta \in C_{\Delta_{\mathbb{N}, \mathbb{R}^{2}, \mathbb{x}}} \) whenever the conditions of the theorem are not satisfied. So, assume that \( \forall \theta > 0, \exists \theta > 0 : m(s) > (\theta) \frac{1}{k} \).

A particular signal \( \xi \in \ell \backslash \ell_2 \) is constructed for which there is an admissible \( \Delta \) such that one has \( (I - \Delta M) \xi \in \ell_2 \). The lack of invertibility of \( (I - \Delta M) \) then follows immediately.

This will be done in two steps. The first step is to construct that signal \( \xi \). The next step is to use this signal to construct a destabilizing perturbation.

Construction of the unbounded signals

Assume that \( M \) is finite-memory. The construction of \( \xi \) proceeds as follows (assume \( b_0 = 0 \) and \( s_0 = 1 \). For any \( n = 1, 2, 3, \ldots \), let \( a_n > 1 \) and \( s_n = a_n s_{n-1} \). Then, for any \( a_n > 1 \), \( \exists s_0 > 0 : m(s_n) > (s_0) \frac{1}{k} \). Let \( a_n = \frac{s_0}{s_0} \geq 1 \) and \( c_n = a_n a_{n-1} \geq 1 \). Then, \( s_n = a_n s_{n-1} = a_n a_{n-1} s_{n-2} = c_n c_n - 1 s_n = 1 s_0 \).

For simplicity, let \( \| M(f) \| = \| M(f) \|_{\ell_0, \infty, \mathbb{x}} \). Let \( f \) be a signal in \( \ell_2 \) of appropriate length for \( i = 1, 2, \ldots \). Choose \( f_n \in \ell_2 \), with \( \| f_n \| = s_n \) and an integer \( N_n > 0 \) such that \( \text{supp}(f_n) = [0, N_n] \) and \( m(s) \) is achieved. This means that

\[
\| M f_n \|_2 > \frac{(s_n)^{\frac{1}{k}}}{k} = \left( \frac{c_n c_n - 1 \cdots c_2}{k} \right)^{\frac{1}{k}}
\]

Let \( t_n = FM(N_n; M) + t_{n-1} - 2 \). Define also \( P_{t_n} - I \xi = (f_1, 0, \ldots, f_0, 0) \). From proposition 2.1, this means that

\[
P_{t_n} - I \xi = (M f_1, x_1, \ldots, M f_n, x_n)
\]

Therefore, we have \( \| P_{t_n} - I \xi \|_2 \) given by

\[
\| P_{t_n} - I \xi \|_2 = s_1 \sqrt{1 + \cdots + (c_n c_n - 1 \cdots c_2)^2}
\]

and \( \| P_{t_n} - I y \|_2 \) given by

\[
\| P_{t_n} - I y \|_2 > \left( \frac{s_1}{k} \right)^{\frac{1}{k}} \sqrt{1 + \cdots + (c_n c_n - 1 \cdots c_2)^2}^{1/2} = \frac{1}{k} \left( 1 \right) \| P_{t_n} - I \xi \|_2 (1 - \epsilon_n)
\]

where

\[
0 < \epsilon_n < 1 - \sqrt{1 + \cdots + (c_n c_n - 1 \cdots c_2)^2}^{1/2} \frac{1 + \cdots + (c_n c_n - 1 \cdots c_2)^2}{1 + \cdots + (c_n c_n - 1 \cdots c_2)^2}^{1/2} < 1
\]

It is easy to see that when \( s_n \rightarrow \infty \), \( c_n \rightarrow \infty \), and therefore \( \epsilon_n \rightarrow 0 \).

Also, \( s_1 > 1 \) and for all \( i = 1, 2, \ldots, n, c_i > 1 \). This means that

\[
\| P_{t_n} - I \xi \|_2 = s_1 \sqrt{1 + \cdots + (c_n c_n - 1 \cdots c_2)} = \sqrt{n}
\]

Therefore, when \( n \rightarrow \infty \), \( \| P_{t_n} - I \xi \|_2 \rightarrow \infty \) and therefore \( \xi \) is unbounded.

Construction of the destabilizing perturbation

Given the signals \( y \) and \( \xi \), we show that a nonlinear, non causal perturbation can be constructed to destabilize the closed-loop system.
Let the signals $y$ and $\xi$ be given as before. $\Delta$ must be constructed such that $\delta(s) < ks^x$ and $\xi = \Delta(y)$. Consider the perturbation defined as follows

$$(\Delta f)(k) = \begin{cases} 0, & \text{if } k < 1, \\ \xi(k-j), & \text{if } j \leq k \leq k_0, \\ 0, & \text{otherwise.} \end{cases}$$

It can be verified that $\Delta$ is a nonlinear and noncausal perturbation. We notice that the maximum amplification occurs when the input signal of $\Delta$ is $\xi$. We also know that

$$\|P_{t_n-1}\xi\|_2 < \left(\frac{1}{1-\epsilon_n}\right)^x k\|P_{t_n-1}\xi\|_2$$

and if take the infimum on the right side of the last inequality over $\epsilon_n$ we actually get

$$\|P_{t_n-1}\xi\|_2 < k\|P_{t_n-1}\xi\|_2$$

which means that $\delta(s) < ks^x$ and therefore $\Delta \in \mathcal{X}_{NC,2}\mathcal{A}$ and maps $y$ to $\xi$.

So, $\Delta$ is constructed in such a way to have $\Delta(y) = \xi - (f_1,0,0,\cdots) = (0,0,f_2,0,f_3,0,f_4,\cdots)$. Now, we just need to show that this is indeed a destabilizing perturbation. If we let $\xi$ be the input to $(I - \Delta M)$ then we have

$$(I - \Delta M)(\xi) = \xi - (M\xi) = (f_1,0,\cdots,0,f_n - f_n,0,\cdots) = (f_1,0,0,0,\cdots) \in \ell_2$$

This implies that the system in figure 1 is not $\ell_2$-stable because it maps a signal in $\ell_2$ to a signal in $\ell \setminus \ell_2$. Therefore, as in the case of the $\ell_\infty$ proof, we conclude that the system is not monotone stable. This completes the proof.

**Comment:** One of the assumptions in the last theorem is $x \leq 1$. The reason that the theorem does not follow for $x > 1$ is because we assumed in the proof that $M$ is finite memory. In fact, if $x > 1$ then $M$ cannot be finite memory. It has to be infinite memory.

**Comment:** In the $\ell_2$ case, the construction of a causal perturbation $\Delta$ instead of a noncausal one, like in the $\ell_\infty$ case where the conditions for stability hold for both NLTV and NLTI causal perturbations, is under investigation.

### 6 Concluding Remarks and Future Work

This paper presented a generalization of the small gain theorem for feedback systems. Sufficient conditions on a system $M$ perturbed by a family of disturbances $\Delta$ were presented. Then, it was shown that those conditions are also necessary in the vector spaces $\ell_\infty$ and $\ell_2$ under appropriate assumptions on the system $M$. For the vector space $\ell_\infty$, those conditions are necessary with either NLTV or NLTI perturbations and for the vector field $\ell_2$, those conditions are necessary with non causal perturbations.

The sufficient conditions are a generalization of the small gain theorem presented in [5]. With this new theorem, the computation work involved in finding the function $\eta_M$ of some system $M$ is simpler since the conditions of the theorem only require that $\eta_M(s) \leq (\xi)^{1/2}$ for big values of $s$. This way, we only need to find $\eta_M$ (or a suitable upper bound of $\eta_M$) for big values of the norm of the input signal of $M$ such that $\eta_M(s) \leq (\xi)^{1/2}$. This is very useful since in most cases it very hard to find $\eta_M(s)$ for all $s > 0$.

In [10] results for the case where the nonlinearities are sector bounded by linear functions are given. As future work, this results can be extended to the case where the nonlinearities are sector bounded by a certain class of monotonic increasing functions. Also, as future work, in the $L_2$ case, a causal perturbation $\Delta$ should be constructed instead of non causal one like in the $\ell_\infty$ case where the conditions for stability hold for both NLTV and NLTI causal perturbations.

### References


