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Controller Design and Implementation for Large-Scale Systems, a Block Decoupling Approach.

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Abstract

In this paper, implementation and controller design issues are broached for a class of large-scale systems. The idea of block decoupling is used as the main tool to achieve great simplification and complexity reduction. Linear input flow constraints are also considered. Two application examples illustrate the main ideas on the paper.

1 Introduction

In this paper, we propose the use of block decoupling to greatly simplify the design and implementation of controllers with specific structures, including hierarchical, for linear large-scale systems with symmetries. Controller structures can also be adopted that facilitate the design in the presence of input linear flow constraints.

We give a compact account of the theory of block decoupling for finite dimensional linear systems. Although notational clarity motivated us to write the paper for time-invariant systems, all the main results extend to the time-variant case. We explore the Kronecker product as a tool to neatly express the coupling structure among subsystems of a linear system of large dimension and use its properties to achieve structural or block decoupling. Certain aspects of the control design of block decoupled systems are broached in section 2; such as decoupled controllers and optimization.

1.1 Notation

We indicate the Kronecker product of matrices as:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & \cdots & a_{nm}\mathbf{B} \end{bmatrix},$$

where $\mathbf{A} \in \mathcal{R}^{n\times m}$, $\mathbf{B} \in \mathcal{R}^{p\times q}$ and $\mathbf{C} \in \mathcal{R}^{np\times mq}$.

The inner product between two vectors is indicated as:

$$\langle \mathbf{x}, \mathbf{y} \rangle_Q = \mathbf{x}^T \mathbf{Q} \mathbf{y},$$

where $\mathbf{Q} > \mathbf{0}$. In this sequence, the norm of a vector is defined as:

$$\|\mathbf{x}\|_Q = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_Q}$$

2 Block decoupling

The problem of system decoupling [4][3] is a well known topic of research, its applications spanning spatially distributed systems as well as large scale systems. Among the many approaches to this problem, we single out the work by [1] that explored the algebraic properties of spatially invariant systems to achieve decoupling. The idea behind it is to figure out what are the subsystems a very complex system is made of and the way they are coupled. Under certain invariance assumptions it is possible to "diagonalize" this coupling structure and come up with a decomposition into systems of smaller dimension. This widely studied topic is strongly related with geometrical notions of invariant subspaces in linear systems.

2.1 Block decoupling of finite dimensional linear systems

Of major importance are the following properties of the Kronecker product:

1. For conformal matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$, the following is a direct consequence of the Kronecker product of matrices:

$$(\mathbf{A}_1 \otimes \mathbf{B}_1)(\mathbf{A}_2 \otimes \mathbf{B}_2) = (\mathbf{A}_1\mathbf{A}_2) \otimes (\mathbf{B}_1\mathbf{B}_2)$$

2. The Kronecker product is a tensor:

$$\begin{aligned} (\alpha\mathbf{A}_1 \otimes \mathbf{B}) + (\beta\mathbf{A}_2 \otimes \mathbf{B}) &= (\alpha\mathbf{A}_1 + \beta\mathbf{A}_2) \otimes \mathbf{B} \\ (\mathbf{B} \otimes \alpha\mathbf{A}_1) + (\mathbf{B} \otimes \beta\mathbf{A}_2) &= \mathbf{B} \otimes (\alpha\mathbf{A}_1 + \beta\mathbf{A}_2) \end{aligned}$$

The following proposition sets the ground for a generalization of the unique representation, of an element of a finite dimensional vector space, as a linear combination of its basis vectors.

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Proposition 1 *Consider a set of linearly independent vectors* $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$, $\mathbf{v}^j \in \mathcal{R}^n$, *then for any other set* $\{\mathbf{z}^1, ..., \mathbf{z}^m\}$, $\mathbf{z}^j \in \mathcal{R}^p$, *follows:*

$$\sum_{j=1}^{m} \mathbf{v}^j \otimes \mathbf{z}^j = \mathbf{0} \Leftrightarrow \mathbf{z}^j = \mathbf{0} \text{ for } j \in [1, m]$$

$$\sum_{j=1}^{m} \mathbf{z}^j \otimes \mathbf{v}^j = \mathbf{0} \Leftrightarrow \mathbf{z}^j = \mathbf{0} \text{ for } j \in [1, m]$$

Remark 1 *If* $\mathbf{v}^l$ *is such that* $\mathbf{\Gamma}\mathbf{v}^l = \lambda_l \mathbf{v}^l$, *then for any matrix* $\mathbf{A}$ *and conformal* $\mathbf{z}$, *the following is true:*

$$(\mathbf{\Gamma} \otimes \mathbf{A})(\mathbf{v}^l \otimes \mathbf{z}) = \lambda_l (\mathbf{v}^l \otimes \mathbf{A}\mathbf{z})$$

The previous equality allow us to show, among other things, the following result:

Theorem 2 *(Block decoupling) Consider the following linear system*

$$\mathbf{x}(t) = \sum_{j=1}^{p} [\mathbf{\Gamma}_j \otimes \mathbf{A}_j]\, \mathbf{x}(t) + [\mathbf{\Gamma}_j \otimes \mathbf{B}_j]\, \mathbf{u}(t), \quad (1)$$

$$\mathbf{y}(t) = \sum_{j=1}^{p} [\mathbf{\Gamma}_j \otimes \mathbf{C}_j]\, \mathbf{x}(t) \quad (2)$$

where $\mathbf{\Gamma}_l \in \mathcal{R}^{m \times m}$ *and* $\mathbf{A}_j \in \mathcal{R}^{n \times n}$. *If there is a set of* m *linearly independent vectors* $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ *such that* $\mathbf{\Gamma}_j \mathbf{v}^l = \lambda_{jl} \mathbf{v}^l$ *for* $(j, l) \in [1, p] \times [1, m]$, *then it is possible to decouple (1) in the form:*

$$\mathbf{x}(t) = \sum_{l=1}^{m} \mathbf{v}^l \otimes \mathbf{x}_D^l(t), \; \mathbf{y}(t) = \sum_{l=1}^{m} \mathbf{v}^l \otimes \mathbf{y}_D^l(t) \quad (3)$$

$$\mathbf{u}(t) = \sum_{l=1}^{m} \mathbf{v}^l \otimes \mathbf{u}_D^l(t), \; \mathbf{y}_D^l(t) = \sum_{j=1}^{p} \lambda_{jl} \mathbf{C}_j \mathbf{x}_D^l(t)$$

where $\mathbf{x}_D^l(t)$ *are solution of:*

$$\dot{\mathbf{x}}_D^l(t) = \left[\sum_{j=1}^{p} \lambda_{jl} \mathbf{A}_j\right] \mathbf{x}_D^l(t) + \left[\sum_{j=1}^{p} \lambda_{jl} \mathbf{B}_j\right] \mathbf{u}_D^l(t) \quad (4)$$

Proof: The result can be easily derived by substituting (3) in (1) and using remark 1. ∎

For a system in the form (1), the decomposition (3) leads to an explicit formula for computing $\mathbf{x}, \mathbf{u}$ and $\mathbf{y}$ as a function of their decoupled versions. In the following theorem, we prove that these transformations are invertible.

Theorem 3 *Consider a set of linearly independent vectors* $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ *and the transformation:*

$$F : \mathbf{z} = \begin{bmatrix} \mathbf{z}^1 \\ \vdots \\ \mathbf{z}^m \end{bmatrix} \mapsto \mathbf{d} = \sum_{j=1}^{m} \mathbf{v}^j \otimes \mathbf{z}^j \quad (5)$$

Then F *has a unique inverse that can be computed from:*

$$\mathbf{z} = (\mathbf{V}^{-1} \otimes \mathbf{I})\, \mathbf{d} \quad (6)$$

where $\mathbf{V} = [\mathbf{v}^1 ... \mathbf{v}^m]$.

Proof: The mapping (5) can be equivalently expressed as:

$$\sum_{j=1}^{m} \mathbf{v}^j \otimes \mathbf{z}^j = (\mathbf{V} \otimes \mathbf{I})\, \mathbf{z}$$

and given that $\mathbf{V}$ is non singular, we can multiply both sides of the previous expression at left by $(\mathbf{V}^{-1} \otimes \mathbf{I})$, to get:

$$(\mathbf{V}^{-1} \otimes \mathbf{I}) \sum_{j=1}^{m} \mathbf{v}^j \otimes \mathbf{z}^j = (\mathbf{V}^{-1}\mathbf{V} \otimes \mathbf{I})\, \mathbf{z} = \mathbf{z}$$

Uniqueness follows directly from proposition 1.

∎

It can also be shown that if the set $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ is orthonormal ($\mathbf{V}$ unitary), then $\mathbf{z}_l$ can be computed from $\mathbf{z}$ by means of the projection:

$$\mathbf{z}_l = [\mathbf{v}^l \otimes \mathbf{I}]^T \mathbf{z} \quad (7)$$

2.2 Control of decoupled systems

Once a system is written in its decoupled form (3)-(4), controllers can be independently designed for the subsystems operating on each of the directions $\mathbf{v}^1, ..., \mathbf{v}^m$. The following remark can be used to achieve the decoupling of optimal control problems:

Remark 2 *For any vectors* $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ *and* $\mathbf{z}, \mathbf{s} \in \mathcal{R}^p$ *and matrices* $\mathbf{\Gamma}_l \in \mathcal{R}^{n \times n}$ *and* $\mathbf{Q}_l \in \mathcal{R}^{p \times p}$, *we get:*

$$\langle \mathbf{x} \otimes \mathbf{z}, \mathbf{y} \otimes \mathbf{s} \rangle_{\sum_{l=1}^{m} \mathbf{\Gamma}_l \otimes \mathbf{Q}_l} = \langle \mathbf{x}, \mathbf{y} \rangle_{\sum_{l=1}^{m} \mathbf{\Gamma}_l} \langle \mathbf{z}, \mathbf{s} \rangle_{\sum_{l=1}^{m} \mathbf{Q}_l},$$

With the notion of orthogonality well defined for the Kronecker product of vectors, we can state the following proposition concerning the decoupling of a class of optimal control problems.

Proposition 4 *Consider a system given by (1) and a set of vectors* $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ *such that the decomposition*

(3)-(4) can be achieved. If the set $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ *satisfies* $\langle \mathbf{v}^q, \mathbf{v}^j \rangle_{\sum_{l=1}^m \Gamma_l} = 0$ *for* $q \neq j$ *and* $\langle \mathbf{v}^j, \mathbf{v}^j \rangle_{\sum_{l=1}^m \Gamma_l} = 1$, *then the solution to the optimal control problem defined by:*

$$\mathbf{u} = \arg\min_{\mathbf{u}} \int_0^T \left\| \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right\|^2_{\sum_{l=1}^m \Gamma_l \otimes \mathbf{Q}_l} dt$$

is given by:

$$\mathbf{u}(t) = \sum_{l=1}^m \mathbf{v}^l \otimes \mathbf{u}_D^l(t), \tag{8}$$

where each $\mathbf{u}_D^l(t)$ *is the solution of:*

$$\mathbf{u}_D^l = \arg\min_{\mathbf{u}_D^l} \int_0^T \left\| \begin{bmatrix} \mathbf{y}_D^l \\ \mathbf{u}_D^l \end{bmatrix} \right\|^2_{\sum_{l=1}^m \mathbf{Q}_l} dt$$

By means of the transformation (6), it can be shown that, under the assumptions of the previous remark, the "total" controller can be written as:

$$\mathbf{u} = \sum_{j=1}^m \mathbf{v}^j \otimes \mathbf{k}_j \left(\left[\mathbf{v}^j \otimes \mathbf{I}\right]^T \left[\sum_{l=1}^m \Gamma_l \otimes \mathbf{Q}_l\right] \mathbf{y}(t) \right) \tag{9}$$

where $\mathbf{k}_l(.)$ are the controllers for each of the directions $\mathbf{v}^1, ..., \mathbf{v}^m$, i.e., $\mathbf{u}_D^l(t) = \mathbf{k}_l(\mathbf{y}_D^l(t))$. In the particular case where $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ is orthonormal, the previous equality simplifies to:

$$\mathbf{u} = \sum_{l=1}^m \mathbf{v}^l \otimes \mathbf{k}_l \left(\left[\mathbf{v}^l \otimes \mathbf{I}\right]^T \mathbf{y}(t) \right),$$

Directly from the invertibility of (5), it follows that if (1) is decoupled as (3)-(4), then the following statements are true:

1. The system (1) is stabilizable (detectable) if and only if each of the m decoupled subsystems is stabilizable (detectable).
2. The "total" controller stabilizes the system (1) if and only if each of the controllers $\mathbf{k}_l(.)$ stabilizes their corresponding subsystem.

2.3 Spatial Invariance

One of the situations where theorem 2 proves to be useful is in the decoupling of spatially invariant systems [2].

Definition: A system described by (1) is spatially invariant if $\mathbf{\Gamma}_j$ are circulant matrices. This definition is motivated by the fact that if $\mathbf{P}$ is a permutation matrix such that $\mathbf{P}\mathbf{e}_i = \mathbf{e}_{i+1 \bmod m}$, then for every circulant $\mathbf{\Gamma}_j$ we have $\mathbf{P}\mathbf{\Gamma}_j\mathbf{P}^{-1} = \mathbf{\Gamma}_j$. By inspection of (1) we see that a spatially invariant system is invariant under state transformations of the type $\widetilde{\mathbf{x}}(t) = (\mathbf{P} \otimes \mathbf{I}_{n \times n})\, \mathbf{x}(t)$.

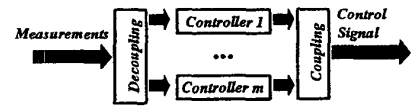

Figure 1: General Control Structure

Remark 3 *For every set of circulant matrices* $\{\mathbf{\Upsilon}_1, ..., \mathbf{\Upsilon}_k\}$, $\mathbf{\Upsilon}_j \in \mathcal{R}^{m \times m}$, *there exist a set of common orthogonal vectors*[1] $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ *such that* $\mathbf{\Upsilon}_j \mathbf{v}^l = \lambda_{jl} \mathbf{v}^l$. *We can use this fact and theorem 2 to state:*

Theorem 5 *Spatially invariant systems can be decoupled in the form (3)-(4).*

3 Controller Structure

According to what was discussed in section 2, after block decoupling, i.e., once we have system (1) expressed in the form (3)-(4), we can design independently the m controllers for each of the "small scale" decoupled systems. The final controller will have the structure depicted in the figure 1, where the "coupling" block implements (8) and "decoupling" represents the inverse transformation.

It is clear that block decoupling may represent great simplification in controller design. Nevertheless, such simplification will not, in general, extend to the implementation of the controller. The worst case situation being the one where all measurements have to be combined in the decoupling block, as well as the controller outputs in the coupling block without any possibility of decentralization.

Although there is no silver bullet for solving these implementation issues, we found that for certain systems the coupling and decoupling blocks may have a rather convenient structure depending on our choice of $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ in (3)-(4). In this choice, Theorem 2 is the only help we have for ruling out sets of vectors which are not suitable.

3.1 Examples of application

We illustrate these ideas by means of two examples; in the first, the coupling block admits a decentralized implementation, while in the second we can implement both blocks by means of a hierarchical structure. Both situations fall in the class of systems where we can also account for a total input flow constraint.

[1] Such vectors result from the Fourier series expansion for sequences of size m and can be computed as $\left[\mathbf{v}^l\right]_q = e^{i2\pi(l/q)}$.

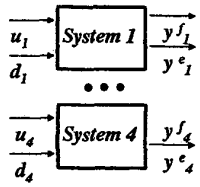

Figure 2: Sketch of the system structure used in the resource allocation setup

3.2 Resource allocation in the presence of an intelligent adversary (Example 1)

In this example we do resource allocation for a linear system with input flow constraints. The problem statement for this problem is the following:

3.2.1 Problem statement: Consider the system as sketched in figure 2. All of the four systems are identically described by the following state space equations:

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \mathbf{B}\begin{bmatrix} u_i \\ d_i \end{bmatrix} \\ \begin{bmatrix} y_i^f \\ y_i^e \end{bmatrix} = \mathbf{C}\mathbf{x}_i \end{cases}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_f & \mathbf{A}_{fe} \\ \mathbf{A}_{ef} & \mathbf{A}_e \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{and } \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

the interaction dynamics is given by:

$$\mathbf{A}_f = \mathbf{A}_e = \begin{bmatrix} -.01 & .03 \\ -.1 & 0 \end{bmatrix}$$

$$\mathbf{A}_{ef} = \mathbf{A}_{fe} = \begin{bmatrix} -.1 & 0 \\ -.1 & 0 \end{bmatrix}$$

Such fourth order systems should be interpreted as independent sectors that comprise two rival second order subsystems, labeled as friendly and enemy. The two state variables of each subsystem represent the amount of "primary resource", which is also the output, and a "secondary resource". As can be inferred from the equations, the following effects are modelled:

1. Primary resources natural decay
2. Primary resources consume secondary resources
3. Secondary resources boost up primary resources
4. Friendly's primary resources destroy enemy's primary and secondary resources and vice versa.
5. The decision variables are u_i. They act as external flux of secondary resources for the friendly subsystem of sector i. The same role is played by d_i regarding the enemy subsystem, and, as such, it is considered as an unknown disturbance.

Regarding controller design, the problem is stated as to design a controller with the following characteristics:

1. Satisfy the input flow constraint $\sum_{i=1}^{4} u_i = 0$. Only transferring resources among neighboring sectors is allowed. We assume the same constraint for the enemy.
2. Minimize the following cost:

$$J = \sup_{\sum_{i=1}^{4} \|d_i\|_2^2 = 1} Q\left(y_i^f - y_i^e\right) + \sum_{i=1}^{3} \delta_{i,i+1} \|r_{i,i+1}\|_2^2$$

where $Q\left(y_i^f - y_i^e\right)$ is a quadratic form, $r_{i,i+1}$ is the flux of secondary resources from sector i to i+1 and $\delta_{i,i+1}$ is the distance between sectors. It should be noted that $r_{i,i+1}$ completely determine u_i and vice versa, provided that the input flow constraint is satisfied. By minimizing J, we choose the strategy that allows the friendly subsystems to engage the enemy and at the same time account for transportation costs.

3.2.2 Controller design: The total system can be put in the form (1) by means of the following state-space equation:

$$\begin{aligned} \mathbf{x}_{total}(t) &= [\mathbf{I}_4 \otimes \mathbf{A}]\, \mathbf{x}_{total}(t) + [\mathbf{I}_4 \otimes \mathbf{B}]\, \mathbf{u}_{total}(t) \\ \mathbf{y}_{total}(t) &= [\mathbf{I}_4 \otimes \mathbf{C}]\, \mathbf{x}_{total}(t) \end{aligned}$$

The above is one of the simplest cases, where the decoupling vectors $\{\mathbf{v}^1, ..., \mathbf{v}^4\}$ can be chosen to be any linearly independent set. The right choice according to the specifications of section 3.2.1 is:

$$\{\mathbf{v}^1, ..., \mathbf{v}^4\} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Each of the resulting four decoupled systems determine the behaviour of the total system along their respective $\mathbf{v}^i$. The motivation behind this choice can be easily understood by viewing each $\mathbf{v}^i$ in the following way:

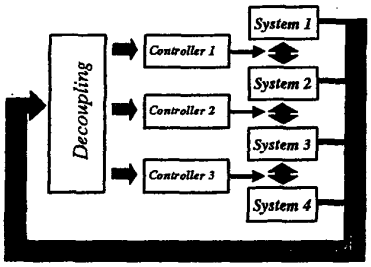

Figure 3: Controller structure for the first setup.

1. $\mathbf{v}^1$ is colinear with the flux constraint and is orthogonal to the remaining $\mathbf{v}^i$. This allow us to isolate the flow constraint so that controller design is unconstrained for the remaining decoupled systems.

2. The output of the controller for the decoupled system associated with $\mathbf{v}^2$ is the required flux of resources from sector 1 to 2. The same principle can be used to explain that $\mathbf{v}^i$ ($i > 1$) is associated with the flux of resources from sector i-1 to i.

In this case, the decomposition (4) is such that all the decoupled systems have the same form:

$$\begin{aligned} \mathbf{x}_D^l(t) &= \mathbf{A}\mathbf{x}_D^l(t) + \mathbf{B}\mathbf{u}_D^l(t) \\ \mathbf{y}_D^l(t) &= \mathbf{C}\mathbf{x}_D^l(t) \end{aligned}$$

Since $\mathbf{u}_{total}$ is completely constrained along $\mathbf{v}^1$, we only need to design 3 controllers. Each controller is designed for a system of dimension 4 (instead of 16) with the above dynamics and using the corresponding decoupled cost.

3.2.3 Controller structure and implementation: The structure of the final controller is the one shown in figure 3. It has the advantage of having a completely decentralized coupling block. The output of the controller i is the flux of resources $r_{i,i+1}$ between neighboring systems i and i+1.

3.2.4 Simulation results: The results for a simulation of example 1 are plotted in figures 4-6, where we decided to present time plots instead of frequency plots in order to illustrate resource transfers easily.

We excited the system by means of the disturbance depicted in figure 4. The shape of the disturbance indicates the enemy's intent of moving resources from sector 1 to sector 4. As can be expected from the model symmetry, the friendly's allocation (control action - plotted in figure 4) tracks the enemy's action. The output (primary resources) is plotted for each sector in figure 5. Figure 6 shows the output of each of the 3 decoupled controllers. By inspecting this plot, one can recognize that the optimal strategy is to shift resources at once to match enemy's allocation. This transfer is done between neighboring sectors at the same time, so that sectors 2 and 3 do not accumulate resources. The slight difference among the plots for controllers 1 through 3 is a result of having assigned different distances between neighboring sectors. Sectors 1 and 2 are closer than sectors 3 and 4, so that controller 1 is allowed to be more aggressive..

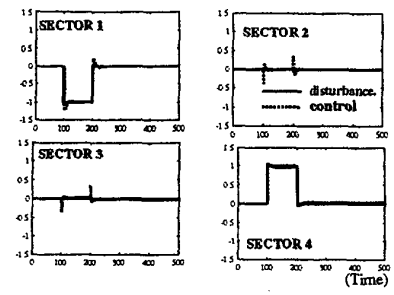

Figure 4: Control and disturbance plots for setup 1.

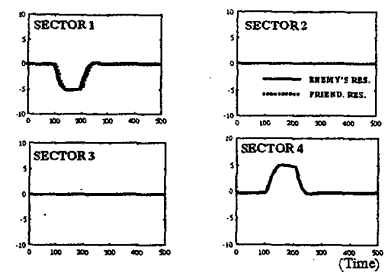

Figure 5: Output for the simulations of setup 1

3.3 Hierarchical Controller Implementation (Example 2)

The previous example is a system that, when expressed in the form (1), leads to matrices $\mathbf{\Gamma}_j$ for which there is more than one set of linearly independent eigenvectors $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$. In this case, one can use this freedom and choose a convenient set. For certain systems it is possible to choose a basis such that each decoupled system is associated with a node of a hierarchical structure. We illustrate this concept by means of a simple example.

3.3.1 Problem statement: Assume we have an array of 8 equal SISO systems that satisfy the state

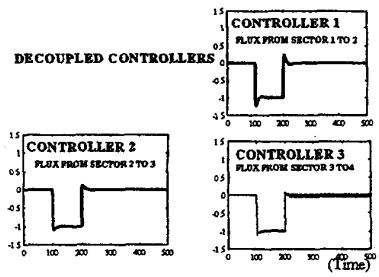

Figure 6: Outputs of the decoupled controllers.

equation $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u + \zeta(t)$, where $\zeta(t)$ accounts for the coupling among systems. Consider that there is a "regional" dynamics $\mathbf{\Gamma}$ such that the global system can be described in the form (1) as:

$$\dot{\mathbf{x}}(t) = [\mathbf{I} \otimes \mathbf{A} + \mathbf{\Gamma} \otimes \mathbf{G}]\,\mathbf{x}(t) + [\mathbf{I} \otimes \mathbf{B}]\,\mathbf{u}(t),$$

where $\mathbf{\Gamma} = \mathbf{I} \otimes \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{x}$ and $\mathbf{u}$ are the vector concatenation of the individual state and control of the 8 subsystems.

3.3.2 Controller design and implementation: In this case, a possible set of eigenvectors would be the one comprising the columns of the matrix in figure 7. This choice of eigenvectors is consistent with a binary partition of the 8 subsystems which is meaningful in resource allocation problems. In this case, $\mathbf{v}^1$ accounts for the "common" behavior or spatial mean value of all systems, $\mathbf{v}^2$ for the transfer of resources between the two groups of systems $\{1,2,3,4\}$ and $\{5,6,7,8\}$. In the next level of partitioning we have $\mathbf{v}^3$ accounting for the transfer of resources between $\{1,2\}$ and $\{3,4\}$, i.e., inside one of the partitions associated with $\mathbf{v}^2$. This nesting relation among the vectors of the basis is represented by the graph of figure 7. Controllers can then be designed for the 8 decoupled systems that operate in the directions $\mathbf{v}^1$ to $\mathbf{v}^8$. In resource allocation problems, where the total amount of resources is constrained by $\sum_{i=1}^{8} [\mathbf{u}(t)]_i = d(t)$, we only need to fix the solution in the direction of $\mathbf{v}^1$ by setting $\mathbf{u}_1(t) = \frac{1}{8} d(t)$. The remaining directions $\mathbf{v}^2$ to $\mathbf{v}^8$ are not affected by the constraint. This example is not that of an uncommon case, for there is a whole class of systems, for which it is possible to model group dynamics in a way that $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$ follows this graph structure.

This structure imposes a hierarchy on the way information flows. Assume we have a decision system (controller) for each of the decoupled systems and that we can measure directly the state $\mathbf{x}(t)$. In such case, we need to process information and get measurements of

$$[\mathbf{v}^1 ... \mathbf{v}^8] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

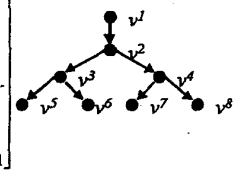

Figure 7: Definition of the eigenvectors $\{\mathbf{v}^1, ..., \mathbf{v}^8\}$ and associated hierarchical structure.

the state of each of the decoupled systems. This can always be done by means of a linear transformation of $\mathbf{x}(t)$. For most sets $\{\mathbf{v}^1, ..., \mathbf{v}^m\}$, this means that the controllers of the decoupled systems must have access to measurements of the entire $\mathbf{x}(t)$, which, for large-scale systems, is not acceptable. But with this hierarchical structure, the state in the direction of, e.g., $\mathbf{v}^8$ can be computed only by measuring $[\mathbf{x}(t)]_7$ and $[\mathbf{x}(t)]_8$. Moving up, $\mathbf{v}^4$ only need to measure $[\mathbf{x}(t)]_7 + [\mathbf{x}(t)]_8$ and $[\mathbf{x}(t)]_5 + [\mathbf{x}(t)]_6$. This way, we may use an information structure based on the graph of figure 1, where a given node adds up his own measurements and makes it available to its upper node as a measurement. Control decisions propagate down the same structure.

4 Conclusions

In this paper an alternative formalism using the Kronecker product was presented to describe the structure of a class of linear systems. It was shown that block decoupled systems have block decoupled controllers and that certain optimal control problems can also be decoupled. A class of linear systems is presented through examples, where simple structures emerge for the flux of information in control applications.

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