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MODEL REDUCTION OF IRREDUCIBLE MARKOV CHAINS

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ABSTRACT

We are interested in developing computational tools for reducing the state space of irreducible Markov chains. As means of decreasing the dimensionality of a given Markov chain we study the concept of aggregation. The approximation error between the original and the reduced order model is captured by a metric that penalizes the asymptotic deviation of the outputs of the two systems. For the case of nearly completely decomposable Markov chains we demonstrate how a decomposition approach can be used to derive a low order model of good fidelity.

1. INTRODUCTION AND BACKGROUND

The motivation of this work comes from problems that are faced by decision making autonomous agents embedded in complex uncertain environments, and are casted in the general framework of Markov Decision Processes (MDP). The associated states spaces are often prohibitive from a computational point of view, thus presenting insurmountable difficulties for the analysis of the task at hand as well as the derivation of optimal policies. In order to handle the intractability, we seek low order representations of the underlying Markovian dynamics in terms of fewer aggregated states.

We consider a time homogenous irreducible Markov chain which evolves on a finite state space denoted by S, where $S = \{s_1, s_2, s_3, ..., s_N\}$. Let ${}^n\pi$ denote the row vector of probability distributions on S at time instant n and let P represent the transition probability matrix. The evolution of the probability distribution vector is given by ${}^{n+1}\pi = {}^n\pi\, P$ and in cases of partial observation we make use of the output equation ${}^n y = {}^n\pi\, C \quad$ where $C \in \mathbf{R}^{N\times m}$.

Under aggregation of the state space we understand a partition of it in disjoint clusters. Given a positive integer $\bar{N}$, typically $\bar{N} \ll N$ one forms a collection of aggregated states $\bar{S} = \{\bar{S}_1, \bar{S}_2, \bar{S}_3, ..., \bar{S}_{\bar{N}}\}$ where $\bar{S}_i \subset S \;\; i \in \{1, ..., \bar{N}\}$, $\bar{S}_i \bigcap \bar{S}_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{\bar{N}} \bar{S}_i = S$. Using the aggregation operator $L : \mathbf{R}^N \rightarrow \mathbf{R}^{\bar{N}}$ where $L_{ij} = 1$ if $s_i \in \bar{S}_j$ and $L_{ij} = 0$ otherwise, one can relate by ${}^n\bar{\pi} = {}^n\pi\, L$ the instantenous probability distributions on $\bar{S}$ and S respectively. Let ${}^n\lambda_{jm}$ denote the conditional probability of the state s_j in cluster $\bar{S}_m$ at instant n i.e., ${}^n\lambda_{jm} = \frac{{}^n\pi_j}{{}^n\bar{\pi}_m}$, then the entries of ${}^n\bar{P}$, the aggregated transition probability matrix at instant n are given by

$$ {}^n\bar{P}_{mk} = \sum_{j\,:\,s_j\in\bar{S}_m} {}^n\lambda_{jm} \sum_{i\,:\,s_i\in\bar{S}_k} P_{ji} \qquad (1) $$

the evolution of the probability distribution for the aggregated system is given by

$$ {}^{n+1}\bar{\pi} = {}^n\bar{\pi}\, {}^n\bar{P} \qquad (2) $$

Note that the Markovian property is preserved, the chain is time inhomogenous though. Computation of the exact value of ${}^n\bar{P}$ requires at each instant a disaggregation step in order to obtain the values of the conditional distributions ${}^n\lambda_{jm}$ in every cluster $\bar{S}_m$. From a computational standpoint this is equivalent with working with the original system, thus exact calculation of the aggregation matrix does not bear any benifit. For a given aggregation operator L we define the compact set of stochastic matrices P_L where

$$ P_L = \{\bar{P} \,:\, \bar{P}_{mk} = \sum_{j:s_j\in\bar{S}_m} \lambda_{jm} \sum_{i:s_i\in\bar{S}_k} P_{ji}\,;\, \lambda_{jm} \in [0,1]\} $$

An equivalent way to equation (2) for describing the evolution of the probability distribution of the aggregated system is given by:

$$ {}^{n+1}\bar{\pi} = {}^1\bar{\pi}\, {}^1\bar{P} ... {}^n\bar{P} \qquad (3) $$

where ${}^i\bar{P} \in P_L \;\; \forall i \in \{1, ..., n\}$. A low order approximation of the original system by a homogenous Markov chain requires a selection of a fixed matrix $\hat{P}$ where $\hat{P} \in P_L$. This is equivalent with fixing the values of conditional probabilities in each cluster $\forall n \in \mathbf{Z}_+$. The approximate dynamics on the aggregated state space will be described by: ${}^{n+1}\hat{\pi} = {}^n\hat{\pi}\, \hat{P}$. The vector ${}^n\hat{\pi}$ is regarded as an approximation to the exact probability distribution ${}^n\bar{\pi}$ on the aggregated state space.

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2. THE MODEL REDUCTION PROBLEM

We denote by M the partially observed Markovian process evolving on S and let $\hat{M}$ denote a process that evolves on $\bar{S}$ which corresponds to the aggregated state space according to according to ${}^{n+1}\hat{\pi} = {}^{n}\hat{\pi}\ \hat{P}$ and ${}^{n}\hat{y} = {}^{n}\hat{\pi}\ \hat{C}$. We assume that $P, \hat{P}$ are irreducible matrices. We will consider a distance function between these two homogenous Markov chains that penalizes deviations between their asymptotic outputs in the sense

$$d(M, \hat{M}) = \|{}^{\infty}y - {}^{\infty}\hat{y}\|_1 \tag{4}$$

The model reduction process requires the solution to the following two subproblems : 1.) Given P, C and the desired $\bar{N}$ establish a selection criterion and an algorithm that produces for each L the corresponding $\hat{P}, \hat{C}$, possibly in an inexpensive manner. 2.) Determine the optimal aggregation operator in the sense $L^o = \arg\min_L d(M, \hat{M})$.
For the particular metric proposed in equation (4) one can compute for each L matrices $\hat{P}, \hat{C}$ such that $d(M, \hat{M}) = 0$. This requires the solution of the Perron eigenvalue problem ${}^{\infty}\pi = {}^{\infty}\pi\ P$ that takes in absence of some special matrix structure $O(N^3)$ operations. In dealing with very large state spaces an algorithm of lower complexity is sought. A suboptimal solution to the model reduction problem for the case of nearly completely decomposable Markov chains is presented in the next section.

3. CASE OF A NEARLY COMPLETELY DECOMPOSABLE MARKOV CHAIN

The concept of a nearly decomposable Markov chain was introduced in [1] for modeling stochastic systems whose dynamics exhibit multiple scales. This characteristic is reflected in a decomposition of the transition probability matrix in the form $P = P^* + \epsilon\Delta$ where P is an irreducible stochastic matrix P^* is a block diagonal stochastic matrix, i.e.

$$P^* = \begin{bmatrix} P_1^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_{\bar{N}}^* \end{bmatrix}$$

with $P_i^*\ \forall i \in \{1, ..., \bar{N}\}$ being irreducible and ϵ denotes a coupling factor that is an otherwise unspecified small number and appears once the entries of Δ are normalized $|\Delta_{ij}| \leq 1\ \forall i, j \in \{1, ..., N\}$. As it was shown in [1] the behavior of such systems can be analyzed in stages. In a first stage one considers each cluster that corresponds to a respective block in P^* independently until a partial equilibrium is achieved. The conditional probabilities in each cluster are determined by solving the $\bar{N}$ decoupled Perron eigenvalue problems

$${}^{\infty}\pi^i = {}^{\infty}\pi^i\ P_i^* \quad i = 1, ..., \bar{N} \tag{5}$$

In a second stage each cluster forms a single aggregated entity and interactions between aggregates lead to the steady state distribution for the system as a whole. One considers the aggregated state space $\bar{S} = \{\bar{S}_1, \bar{S}_2, ..., \bar{S}_{\bar{N}}\}$ where $\bar{S}_i$ contains the states corresponding to the block P_i^*. Let $\hat{P}$ denote the matrix obtained by substituting the conditional probabilities obtained by equation (5) in equation (1). By solving ${}^{\infty}\hat{\pi} = {}^{\infty}\hat{\pi}\ \hat{P}$ for the Perron eigenvector of $\hat{P}$, one obtains the following approximation to the steady state distribution of P:

$${}^{\infty}\tilde{\pi} = \left[\ {}^{\infty}\hat{\pi}_1\ {}^{\infty}\pi^1\ \ \ldots\ \ {}^{\infty}\hat{\pi}_{\bar{N}}\ {}^{\infty}\pi^{\bar{N}}\ \right]$$

An asymptotic result on the associated approximation error can be found in [2], in particular $\|{}^{\infty}\pi - {}^{\infty}\tilde{\pi}\|_1 = O(\epsilon)$. The methods described [1], [2] can be used to derive a reduced order model for a nearly completely decomposable Markov chain in an analogous fashion provided that the considered metric penalizes only deviations in the steady state as it is the case in equation (4). Without loss of generality let $C = I$, then by using the aggregation operator L compatible with the block structure of P^* as described above one obtains a reduced order Markov chain $\hat{P}$ with $\bar{N}$ states and by setting

$$\hat{C} = \begin{bmatrix} {}^{\infty}\pi^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & {}^{\infty}\pi^{\bar{N}} \end{bmatrix}$$

it is ensured that $d(M, \hat{M}) = O(\epsilon)$

4. FUTURE WORK

As part of future work we would like to derive sharp bounds for the approximation error of the above model reduction procedure. Clustering algorithms are to be considered so that an arbitrary matrix can be brought to the form $P = P^* + \epsilon\Delta$ whith ϵ being minimized. The problem of determining the coarsest aggregation for a given error bound is also to be addressed, in particular we would like to be able to calculate for a given degree of accuracy $\epsilon > 0$, the smallest $\bar{N} : \exists L$ with $d(M, \bar{M}) \leq \epsilon$. Finally we would like to extend the model reduction procedure to the case of general irreducible Markov chains and for various metrics that capture also transient effects.

5. REFERENCES

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Bilinear Model Approximation for a Class of Regionally Stable Uncertain Nonlinear Systems[1]

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Abstract

This paper proposes a convex approach to the model approximation problem for a class of regionally stable uncertain nonlinear systems. More specifically, we determine an extended bilinear system which approximates in a given region of the state space the input-to-output dynamics of the nonlinear system with time-varying parameters. To this end, we use a suitable parametrization of the Lyapunov matrix in order to obtain convex model design conditions in terms of linear matrix inequalities (LMIs). The proposed approach is also extended to the model reduction case without rank constraints.

1 INTRODUCTION

The last decades or so have witnessed active research in the area of robust model approximation and order reduction for large scale linear systems, see e.g. [1] and references therein. The basic idea is to determine a linear low order system that approximates the dynamics of a high-order (possibly uncertain) linear system with a (small enough) guaranteed upper-bound on the approximation error. A more realistic situation is the model approximation of nonlinear systems providing simplified procedures for performance analysis and design. In general, approximate versions of nonlinear dynamics are obtained with the use of (uncertain) linear models such as the works of Chao & Fitzsimons in [2], Lawrence in [3] and Glad et al in [4].

On the other hand, starting from the observation that a nonlinear system can be approximated by a Taylor expansion around an equilibrium point an alternative modelling for the identification (and also approximation) of nonlinear systems is the class of bilinear systems [5] and its extensions (second degree [6] and extended bilinear systems [7]). The usual bilinear approximation consists of concentrating all nonlinearities in the input dynamics neglecting the quadratic and high orders terms of the Taylor expansion. This technique achieves good results only if the nonlinearities are concentrated in the input dynamics. A natural extension of the bilinear representation is to consider the quadratic terms of the state vector in the Taylor expansion leading to the extended bilinear systems, i.e. the nonlinear map

$$y(t) = c(x(t), u(t)),\ x \in \mathbb{R}^n,\ \dot{x}(t) = a(x(t), u(t)),$$

is approximated by the map

$$\begin{aligned} \eta(t) &= G(\sigma)\sigma(t) + H(\sigma)u(t), \\ \dot{\sigma}(t) &= E(\sigma)\sigma(t) + F(\sigma)u(t), \end{aligned}$$

where the matrices $G(\sigma), H(\sigma), E(\sigma)$ and $F(\sigma)$ are affine functions of $\sigma \in \mathbb{R}^m$. The above class of systems offers a good compromise between accuracy and simplicity of the model, since we obtain a better approximation than the bilinear one and we can use the standard linear matrix inequality (LMI) framework for performance analysis and control [8].

The purpose of this paper is to devise a technique for approximating a regionally stable uncertain nonlinear system with bounded inputs by a time-invariant extended bilinear systems with the same order of the nonlinear system ($n = m$) in an $\mathcal{H}_\infty$ sense. To this end, we apply a suitable parametrization of the Lyapunov matrix obtaining convex conditions in terms of LMIs that assure the minimization of the (worst-case) error signal $e(t) = y(t) - \eta(t)$. We then extend the proposed approach for model reduction, i.e. $m < n$, by imposing a constraint on the structure of the Lyapunov matrix.

The rest of this paper is as follows. Section 2 formalizes the problem of concern and Section 3 presents some preliminary results. The model approximation conditions are stated in Section 4 and Section 5 extends these results for model reduction. Finally, Section 6 ends the paper.

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