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Finite Approximations of Switched Homogeneous Systems for Controller Synthesis

Danielle C. Tarraf, *Member, IEEE*,
Alexandre Megretski, *Senior Member, IEEE*, and
Munther A. Dahleh, *Fellow, IEEE*

***Abstract*—In this note, we demonstrate the use of a control oriented notion of finite state input/output approximation to synthesize correct-by-design controllers for hybrid plants under sensor limitations. Specifically, we consider the problem of designing stabilizing switching controllers for a pair of unstable homogeneous second order systems with binary output feedback. In addition to yielding a deterministic finite state approximate model of the hybrid plant, our approach allows one to efficiently establish a useable upper bound on the quality of approximation, and leads to a discrete optimization problem whose solution immediately provides a certified finite state controller for the plant. The resulting controller consists of a deterministic finite state observer and a corresponding full state feedback control law.**

***Index Terms*—Binary output feedback, certified controller, finite memory controller, finite state approximation, hybrid systems, input/output approximation.**

I. Introduction

Finite approximations of hybrid plants have been used to simplify complex synthesis problems that cannot be handled well by traditional methods. Early research explored "qualitative" models, non-deterministic finite automata whose output or input/output behavior contains that of the original hybrid system [17]. In [22], qualitative models of systems with quantized outputs were proposed in conjunction with supervisory control theory [23] to design controllers meeting the desired specifications.

Inspired by the theory of bisimulation in concurrent processes [19], [20], another line of research explored finite bisimulation abstractions of hybrid systems. It was soon recognized that the classes of systems admitting finite bisimuations are limited [10], [14]. Moreover even when they do exist, these abstractions tend to be prohibitively large (in number of states, see for instance [2]) rendering the approach computationally inefficient. Recent focus has thus turned to variants of this approach employing weaker notions of abstraction and less stringent metrics [7], [26], [27]. Controller synthesis typically requires designing a finite state supervisory controller for the finite abstraction and subsequently refining it to yield a certified hybrid controller [28]. Alternative design procedures inspired by linear temporal logic (LTL) model checking have also been proposed [13]. Assumptions on the underlying dynamics, such as piecewise-affine [9] or incrementally stable [8] plants, are often required to ensure existence of the finite abstraction. To date, the research in this area has largely focused on the full state feedback problem: State estimation and observer design problems have not been seriously considered.

In this note, we demonstrate a new,[1] robust control inspired approach to input/output approximation, allowing us to elegantly address the binary output feedback control synthesis problem while simultaneously avoiding some of the drawbacks of the above methods. Indeed our approach yields a deterministic finite state approximate model of the plant, allows one to efficiently bound the quality of approximation, and leads to a tractable optimization problem whose solution immediately provides a certified finite state controller for the plant. The resulting controller consists of a deterministic finite state observer and a corresponding full state feedback control law.

Specifically, we consider the problem of designing stabilizing switching controllers for a pair of unstable homogeneous second order systems under binary output feedback. This problem was chosen as it remains challenging even while several of its simpler formulations are well understood [4], [15]. Necessary and sufficient conditions for stability of switched second order homogeneous systems [6] and a Lyapunov based approach for designing stabilizing full state feedback controllers [11] have been demonstrated. In the analog full state feedback LTI case, the existence of a Hurwitz convex combination (A_{eq}) of a given pair of unstable state matrices is necessary [5] and sufficient [34] for the existence of a quadratically stabilizing switching controller. When A_{eq} has a real eigenvalue, a quadratic switching surface and state dependent variable structure control law [12] can be designed to stabilize the system [34]. Lyapunov based approaches have been extended to time-sampled [3], [16] as well as output feedback setups where the output is a linear function of the state [25]. In contrast, *fixed binary sensors* present a difficult state estimation and control synthesis problem that began to receive attention recently [1], [18], [21], [24]; The use of finite models remains minimally explored.

Organization: The problem is stated in Section II. Algorithms for constructing finite approximations of the plant and for computing approximation error bounds are presented in Section III. Design of the stabilizing controller is addressed in Section IV, with the technical proofs deferred to the Appendix. An illustrative example follows in Section V.

Notation: $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the reals, non-negative reals and non-negative integers, respectively. $|I| = |b-a|$ denotes the length of $I = [a, b)$. v' denotes the transpose of $v \in \mathbb{R}^n$ and $\|v\| = \sqrt{v'v}$ its Euclidean norm. $A^{\mathbb{Z}_+}$ denotes the set of all infinite sequences over A,

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D. C. Tarraf is with the Department of Electrical and Computer Engineering, Johns Hopkins University, Baltimore, MD 21218 USA (e-mail: dtarraf@jhu.edu).

A. Megretski and M. A. Dahleh are with the Electrical Engineering and Computer Science Department and the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: ameg@mit.edu; dahleh@mit.edu).

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[1] Earlier versions of this approximation approach were explored in [29]–[32].

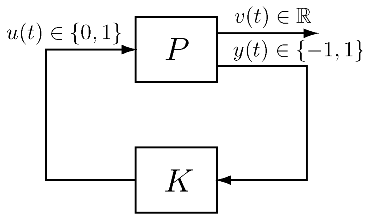

Fig. 1. Closed loop system.

$\mathbf{a}$ denotes an element of $A^{\mathbb{Z}_+}$. $A \times B$ denotes the Cartesian product of A and B, $card(A)$ denotes the cardinality of A, and A^B denotes the set of all functions from B to A. For $f : A \to B$, $g : B \to C$, and $A' \subset A$, $f|_{A'} : A' \to B$ denotes the restriction of f to A' and $g \circ f : A \to C$ denotes the composition of f and g. For $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, $f \leq g$ signifies $f(a) \leq g(a), \forall a \in A$.

II. Problem Statement

Given a discrete-time plant P described by

$$\begin{aligned} x(t+1) &= f_{u(t)}\left(x(t)\right) \\ y(t) &= sign\left(c'x(t)\right) \\ v(t) &= log\left(\frac{\|x(t+1)\|}{\|x(t)\|}\right) \end{aligned} \tag{1}$$

where $t \in \mathbb{Z}_+$, $x(t) \in \mathbb{R}^2$, $v(t) \in \mathbb{R}$, $u(t) \in \mathcal{U} = \{0,1\}$, and $y(t) \in \mathcal{Y} = \{-1,1\}$; y is assigned the value $+1$ in one quadrant when $c'x = 0$ and -1 otherwise. $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c \in \mathbb{R}^2 \setminus \{0\}$ is given. $f_{0,1} : \mathbb{R}^2 \to \mathbb{R}^2$ are given continuous functions, homogeneous with degree 1: That is, $f_u(\alpha x) = \alpha f_u(x)$ for all $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^2$. The goal is to design a controller K such that the closed loop system (Fig. 1) satisfies

$$\sup_{T \geq 0} \sum_{t=0}^{T} \left(v(t) + R\right) < \infty \tag{2}$$

for some $R > 0$, for all initial conditions of P. Satisfying this performance objective guarantees that the state of the closed loop system globally "exponentially" converges[2] to the origin at a rate not less than R. The assumption is that neither subsystem has a globally stable equilibrium point at the origin, otherwise the problem is trivial.

III. A Finite State Approximation of the Plant

A. *Construction of the Nominal Model*

In this section, we describe the construction of a deterministic finite state machine (DFM) input/output approximation of plant (1) and performance objective (2). A DFM is a discrete-time dynamical system described by state transition (3) and output (4) equations

$$q(t+1) = f\left(q(t), p(t)\right) \tag{3}$$

$$z(t) = g\left(q(t), p(t)\right) \tag{4}$$

with $t \in \mathbb{Z}_+$, state $q(t) \in \mathcal{Q}$, input $p(t) \in \mathcal{P}$, and output $z(t) \in \mathcal{Z}$, where $\mathcal{Q}$, $\mathcal{P}$ and $\mathcal{Z}$ are finite sets. A set of initial states $\mathcal{Q}_o \subset \mathcal{Q}$ may be specified.

[2]Note that (2) can be equivalently rewritten as

$$\|x(t)\| \leq k\left(x(0)\right) 10^{-Rt} \|x(0)\|, \quad \forall t > 0, \quad x(0) \in \mathbb{R}^2$$

where $k(x(0)) = 10^{S(x(0))-R}$ and $S(x(0)) = \sup_{T \geq 0} \sum_{t=0}^{T} (v(t) + R)$. The notion of 'exponential' convergence is thus slightly weaker than the standard one since we do not require $k(x(0))$ to be uniformly bounded.

In our construction, we exploit a known property of homogeneous systems evident in polar coordinates (r, θ), namely that the angular coordinate θ and outputs y and v evolve independently of the radial coordinate r: The system state relevant to our problem effectively evolves on the unit circle. Our construction involves 3 steps.

Step 1) Partition the unit circle into intervals $I_1, \ldots, I_{2n}$, where $I_i = [\alpha_i, \alpha_{i+1})$ for some sequence of angles $\alpha_1 < \ldots < \alpha_{2n+1}$ satisfying $\alpha_1 = tan^{-1}(-c_1/c_2)$, $\alpha_1 \in [0, \pi)$, $\alpha_{n+1} = \alpha_1 + \pi$, $\alpha_{2n+1} = \alpha_1 + 2\pi$. The number and choice of angles is a design parameter. Let S_k be the set of all 'k adjacent intervals': That is, $S_1 = \{I_1, I_2, \ldots, I_{2n}\}$, $S_2 = \{I_1 \cup I_2, I_2 \cup I_3, \ldots, I_{2n} \cup I_1\}$, and so on with $S_{2n} = \{I_1 \cup I_2 \cup \ldots \cup I_{2n}\}$. Construct a DFM $\hat{M}$ with state set[3] $\hat{\mathcal{Q}} = S_1 \cup S_2 \cup \ldots \cup S_{2n}$ and dynamics given by

$$\begin{aligned} q(t+1) &= \hat{f}\left(q(t), u(t), \tilde{y}(t)\right) \\ \hat{y}(t) &= \hat{g}\left(q(t)\right) \\ \hat{v}(t) &= \hat{h}\left(q(t), u(t)\right) \end{aligned} \tag{5}$$

with $\hat{f} : \hat{\mathcal{Q}} \times \mathcal{U} \times \mathcal{Y} \to \hat{\mathcal{Q}}$ and $\hat{g} : \hat{\mathcal{Q}} \to \mathcal{Y}$ defined by

$$\hat{f}(q, u, \tilde{y}) = \bigcup_{i \in \mathcal{I}^u_{\mathcal{P}_{\tilde{y}}(q)}} [\alpha_i, \alpha_{i+1}),$$

$$\hat{g}(q) = \begin{cases} 1 & \text{if } q = \mathcal{P}_1(q) \text{ or } |\mathcal{P}_1(q)| \geq |\mathcal{P}_{-1}(q)| \\ -1 & \text{if } q = \mathcal{P}_{-1}(q) \text{ or } |\mathcal{P}_1(q)| < |\mathcal{P}_{-1}(q)| \end{cases}$$

and $\hat{h} : \hat{\mathcal{Q}} \times \mathcal{U} \to \mathcal{V}$ ($\mathcal{V}$ is a discrete subset of $\mathbb{R}$) defined by

$$\hat{h}(q, u) = \sup_{\theta \in q} \left[log\left(\|f_u\left(\beta(\theta)\right)\|\right)\right]$$

where $f_u(.) = \begin{bmatrix} f_{u1}(.) \\ f_{u2}(.) \end{bmatrix}$, $\beta(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$\mathcal{I}^u_q = \{i \in \{1, \ldots, 2n\} | q^u_+ \cap [\alpha_i, \alpha_{i+1}) \neq \emptyset\}$

$q^u_+ = tan^{-1}(f_{u2}(\beta(q))/f_{u1}(\beta(q)))$, and $\mathcal{P}_i(q), j \in \{-1, 1\}$, equals $q \cap [\alpha_1, \alpha_{n+1})$ if $sign(c'\beta((\alpha_1 + \alpha_{n+1})/2)) = j$ and $q \cap [\alpha_{n+1}, \alpha_{2n+1})$ otherwise.

Step 2) Constrain the initial state of $\hat{M}$ to $q_o = [\alpha_1, \alpha_{2n+1})$, corresponding to the whole unit circle. The *actual* states of $\hat{M}$, $\mathcal{Q} \subset \hat{\mathcal{Q}}$, are then those states reachable from q_o. The problem of computing $\mathcal{Q}$ can be recast as either one of two well-studied problems: (i) A one-to-all network shortest path problem, which can be efficiently solved (polynomial time in n) using any of the available shortest path algorithms (Dijkstra's, Bellman-Ford, ...), or (ii) the problem of computing the accessible states of an automaton, which can be efficiently solved by constructing the transition tree of the automaton. The dynamics of $\hat{M}$ are now given by

$$\begin{aligned} q(t+1) &= f\left(q(t), u(t), \tilde{y}(t)\right) \\ \hat{y}(t) &= g\left(q(t)\right) \\ \hat{v}(t) &= h\left(q(t), u(t)\right) \end{aligned} \tag{6}$$

where $f = \hat{f}|_{\mathcal{Q} \times \mathcal{U} \times \mathcal{Y}}$, $g = \hat{g}|_{\mathcal{Q}}$, $h = \hat{h}|_{\mathcal{Q} \times \mathcal{U}}$.

Step 3) Construct a DFM M with inputs $u(t) \in \mathcal{U}$, $w(t) \in \mathcal{W} = \{0, 1\}$, outputs $u(t) \in \mathcal{U}$, $\hat{v}(t) \in \mathcal{V}$, $\tilde{y}(t) \in \mathcal{Y}$, and internal structure shown in Fig. 2, with $\phi(\hat{y}, w) = \hat{y}$ when $w = 0$ and $\phi(\hat{y}, w) = -\hat{y}$ when $w = 1$.

Theorem 1: Consider a plant P as in (1), a performance objective (2), and a DFM M constructed according to the 3 steps proposed above. There exists a system $\Delta \subset \mathcal{U}^{\mathbb{Z}_+} \times \mathcal{W}^{\mathbb{Z}_+}$, and functions $\rho_\Delta : \mathcal{U} \to$

[3]$\hat{\mathcal{Q}}$ consists of $2n(2n-1)+1$ distinct elements and represents the set of all *potential* states of the approximate model.

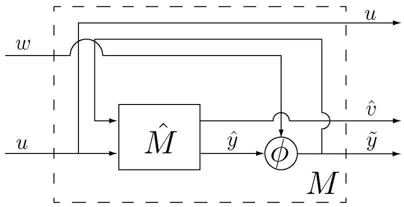

Fig. 2. Internal structure of M, the DFM approximation of P.

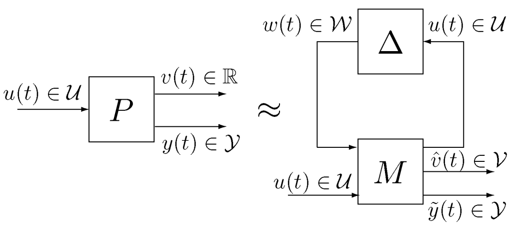

Fig. 3. Plant P, its DFM approximation M, and approximation error Δ.

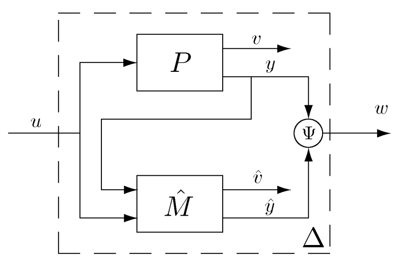

Fig. 4. Approximation error Δ of P and $\hat{M}$.

$\mathbb{R}_+$ and $\mu_\Delta : \mathcal{W} \to \mathbb{R}_+$, not identically zero, such that the feedback interconnection (M, Δ) shown in Fig. 3 satisfies:

1) For any input $\mathbf{u} \in \mathcal{U}^{\mathbb{Z}_+}$ and any initial condition of P:
   a) Outputs $\mathbf{y} \in \mathcal{Y}^{\mathbb{Z}_+}$ of P and $\tilde{\mathbf{y}} \in \mathcal{Y}^{\mathbb{Z}_+}$ of (M, Δ) are identical.
   b) Outputs $\mathbf{v} \in \mathbb{R}^{\mathbb{Z}_+}$ of P and $\hat{\mathbf{v}} \in \mathcal{V}^{\mathbb{Z}_+}$ of (M, Δ) satisfy

$$\hat{v}(t) \geq v(t), \quad \text{for all} \quad t \in \mathbb{Z}_+. \tag{7}$$

2) There exists a constant $\gamma \geq 0$ such that every pair of signals $(\mathbf{u}, \mathbf{w}) \in \Delta$ satisfies

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta\left(u(t)\right) - \mu_\Delta\left(w(t)\right) > -\infty. \tag{8}$$

Proof: Consider system Δ shown in Fig. 4, with

$$\Psi(y, \hat{y}) = \begin{cases} 0 & \text{if } y = \hat{y} \\ 1 & \text{otherwise.} \end{cases}$$

It can be verified by direct inspection that 1(a) is satisfied. Moreover, the choice of $\gamma = 1$ and functions ρ_Δ and μ_Δ defined by $\rho_\Delta(u) = 1$ and $\mu_\Delta(w) = w$ satisfies (8). Let q denote the state of M, and let θ denote the angular coordinate of P when (1) is rewritten in polar coordinates. It follows from the construction of $\hat{M}$ that $\theta(t) \in q(t) \Rightarrow \hat{v}(t) \geq v(t)$ and $\theta(t+1) \in q(t+1)$, for any choice of t. Since $\theta(0) \in [0, 2\pi) = [\alpha_1, \alpha_{2n+1}) = q_o = q(0)$, we have $\hat{v}(0) \geq v(0)$ and $\theta(1) \in q(1)$. Statement 1(b) thus follows by induction on t. ■

Remark 1: The DFM M constructed according to the proposed 3 step procedure is thus a finite input/output approximation of plant P (1) and performance objective (2). The internal structure of approximation error Δ, where the output of P is fed back to $\hat{M}$, differs from the traditional stable LTI model reduction setting in which the approximation error is simply the difference of the two systems. This structure is needed because the plant is not stable: Thus, unless $\hat{M}$ is allowed to 'estimate' the initial condition of P, there is no hope of satisfying (8). Also note that by construction, there is no direct feedthrough from input $\tilde{y}$ to output $\hat{y}$ in $\hat{M}$. This ensures that the outputs y of P and $\hat{y}$ of $\hat{M}$ cannot be trivially matched.

B. Description of the Approximation Error

The "gain" γ^* of system Δ (the infimum of the set of values of γ for which (8) holds) represents the fraction of time (computed over an infinite window) that the outputs y of P and $\hat{y}$ of $\hat{M}$ disagree in the worst-case scenario. A smaller gain thus indicates a better approximation. Computing γ^* is difficult, since Δ is a complex system with hybrid (analog/discrete) state-space. Nonetheless, our construction of $\hat{M}$ allows us to efficiently establish an *upper bound* for γ^*.

Theorem 2: Consider function $d : \mathcal{Q} \to \{0, 1\}$ defined by $d(q) = 0$ if $q = \mathcal{P}_1(q)$ or $q = \mathcal{P}_{-1}(q)$, and $d(q) = 1$ otherwise. If there exists a $\gamma > 0$ and a function $V : \mathcal{Q} \to \mathbb{R}$ such that

$$V\left(f(q, u, y)\right) - V(q) \leq \gamma \rho_\Delta(u) - d(q) \tag{9}$$

holds for all $q \in \mathcal{Q}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$, then Δ satisfies (8) for that choice of γ.

Proof: By summing up (9) along any state trajectory of $\hat{M}$ from $t = 0$ to $t = T$, we get

$$\begin{aligned} \sum_{t=0}^{T} \gamma \rho_\Delta\left(u(t)\right) - d\left(q(t)\right) &\geq V\left(q(T+1)\right) - V\left(q(0)\right) \\ &\geq \min_{q_1, q_2} \left(V(q_1) - V(q_2)\right) > -\infty \end{aligned}$$

Hence, we have

$$\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho_\Delta\left(u(t)\right) - d\left(q(t)\right) > -\infty.$$

Once again, let θ denote the angular coordinate of P and let q denote the state of $\hat{M}$. By construction, $\theta(t) \in q(t)$ for all t. Thus, when $q(t) = \mathcal{P}_1(q(t))$ or $q(t) = \mathcal{P}_{-1}(q(t))$, $y(t) = \hat{y}(t)$ and $w(t) = 0$, else $w(t) \in \{0, 1\}$. Hence $w(t) \leq d(q(t))$ for all t and all feasible signals of Δ satisfy (8). ■

An upper bound for γ^* can thus be computed by solving a linear program with $N + 1$ decision variables and $4N$ inequality constraints ($N = card(\mathcal{Q})$) in which we minimize γ such that (9) holds for all $q \in \mathcal{Q}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$. This approach, while computationally efficient, results in conservative gain bounds for two reasons: First, it assumes that an error occurs every time it can. Second, it assumes that every $(\mathbf{u}, \mathbf{y}) \in \mathcal{U}^{\mathbb{Z}_+} \times \mathcal{Y}^{\mathbb{Z}_+}$ is a valid input sequence for $\hat{M}$, not the case here as $\mathbf{y}$ is an output of P corresponding to input $\mathbf{u}$.

IV. Controller Design

Consider the following two controller synthesis problems:

Problem 1: Given a plant P as in (1), design a controller $K \subset \mathcal{Y}^{\mathbb{Z}_+} \times \mathcal{U}^{\mathbb{Z}_+}$ such that feedback interconnection (P, K) in Fig. 1 satisfies (2) for some $R > 0$, for any initial condition of P. □

Problem 2: Given a plant P as in (1), an approximation $M \subset (\mathcal{W}^{\mathbb{Z}_+} \times \mathcal{U}^{\mathbb{Z}_+}) \times (\mathcal{U}^{\mathbb{Z}_+} \times \mathcal{V}^{\mathbb{Z}_+} \times \mathcal{Y}^{\mathbb{Z}_+})$ of P constructed as described in Section III-A, and a verified gain bound γ_o for the corresponding approximation error Δ. Design a full state feedback control law $\varphi : \mathcal{Q} \to \mathcal{U}$ such that the interconnection (M, Δ, φ) shown in Fig. 5 satisfies the auxiliary robust performance objective

$$\sup_{T \geq 0} \sum_{t=0}^{T} \left(\hat{v}(t) + R\right) < \infty \tag{10}$$

for some $R > 0$, for any $\Delta \in \mathbf{\Delta}_{\gamma_o}$ where

$$\mathbf{\Delta}_{\gamma_o} = \left\{ \Delta \subset \mathcal{U}^{\mathbb{Z}_+} \times \mathcal{W}^{\mathbb{Z}_+} \middle| \Delta \text{ satisfies (8) for } \gamma = \gamma_o \right\}. \quad \square$$

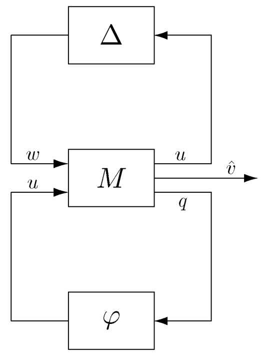

Fig. 5. Robust full state feedback control design setup.

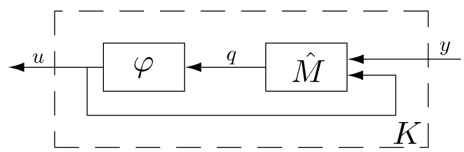

Fig. 6. Internal structure of the finite state stabilizing controller K.

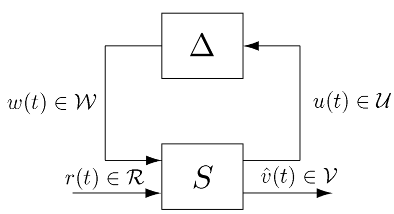

Fig. 7. Setup for the "small gain" theorem.

Any solution to Problem 2 immediately provides a solution to Problem 1, the original problem of interest: Let φ be a full state feedback law such that (M, Δ, φ) satisfies (10) for some $R > 0$, for all $\Delta \in \mathbf{\Delta}_{\gamma_o}$. To ensure that (2) holds for (P, K), it is sufficient to construct a controller K that is identical to the subsystem with input y and output u in interconnection (M, Δ, φ). The resulting controller K, shown in Fig. 6, thus consists of $\hat{M}$, a DFM "observer" for the plant and φ, a corresponding full state feedback control law.

The robust switching law φ in Problem 2 can be designed using dynamic programming techniques and a "small gain" argument. The following Theorem is adapted from Theorem 1 and Remark 4 in [33].

Theorem 3: (*A 'Small Gain' Theorem*) Consider the feedback interconnection of two systems S and Δ as in Fig. 7. If S satisfies

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_S\left(r(t), w(t)\right) - \mu_S\left(\hat{v}(t), u(t)\right) > -\infty \tag{11}$$

for some $\rho_S : \mathcal{R} \times \mathcal{W} \to \mathbb{R}$, $\mu_S : \hat{\mathcal{V}} \times \mathcal{U} \to \mathbb{R}$, and Δ satisfies

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_\Delta\left(u(t)\right) - \mu_\Delta\left(w(t)\right) > -\infty$$

for some $\rho_\Delta : \mathcal{U} \to \mathbb{R}$, $\mu_\Delta : \mathcal{W} \to \mathbb{R}$, then (S, Δ) satisfies

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho\left(r(t)\right) - \mu\left(\hat{v}(t)\right) > -\infty$$

for $\rho : \mathcal{R} \to \mathbb{R}$, $\mu : \hat{\mathcal{V}} \to \mathbb{R}$ defined by

$$\rho(r) = \sup_{w \in \mathcal{W}} \left\{\rho_S(r, w) - \tau \mu_\Delta(w)\right\},$$

$$\mu(\hat{v}) = \inf_{u \in \mathcal{U}} \left\{\mu_S(\hat{v}, u) - \tau \rho_\Delta(u)\right\}$$

for any $\tau > 0$. □

TABLE I
DATA FOR ILLUSTRATIVE EXAMPLE ($k_1 = -3$)

n	N	T_S	γ	R	p
5	39	0.6283	0.75	0.0160	10
10	107	0.3142	0.625	0.0267	20
15	207	0.2094	0.5833	0.0195	28
20	331	0.1571	0.5625	0.0152	39

It follows from Theorem 3 that (10) can be achieved by designing $\varphi : \mathcal{Q} \to \mathcal{U}$ such that $S = (M, \varphi)$ satisfies (11) with $\rho_S(r, w) = \tau \mu_\Delta(w) - R$ and $\mu_S(\hat{v}, u) = \hat{v} + \tau \gamma_o \rho_\Delta(u)$ for some $R > 0, \tau > 0$. Exogenous input r can be assumed to be constant here, representing the desired rate of convergence. With (10) reformulated as a design objective for $S = (M, \varphi)$, design of the switching law φ reduces to solving a min-max optimization problem. Techniques inspired by dynamic programming can be used, as shown in Theorem 4. In this setting, J and $\mathbb{T}$ are the "cost-to-go" function and the "dynamic programming" operator, respectively. Value iteration is used to solve for J, and φ is then simply the optimizing argument. A complete proof is presented in the Appendix.

Theorem 4: Consider a DFM M with state transition equation $q(t+1) = f(q(t), u(t), w(t))$, and let $\sigma : \mathcal{Q} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}$ be given. The following three statements are equivalent:

a) There exists a $\varphi : \mathcal{Q} \to \mathcal{U}$ such that the closed loop system (M, φ) satisfies

$$\inf_{T \geq 0} \sum_{t=0}^{T} \sigma\left(q(t), \varphi\left(q(t)\right), w(t)\right) > -\infty. \tag{12}$$

b) There exists a function $J : \mathcal{Q} \to \mathbb{R}_+$ such that the inequality

$$J(q) \geq \mathbb{T}\left(J(q)\right) \tag{13}$$

holds for any $q \in \mathcal{Q}$, for $\mathbb{T} : \mathbb{R}^{\mathcal{Q}} \to \mathbb{R}^{\mathcal{Q}}$ defined by

$$\mathbb{T}\left(J(q)\right) = \min_{u \in \mathcal{U}} \max_{w \in \mathcal{W}} \left\{-\sigma(q, u, w) + J\left(f(q, u, w)\right)\right\}. \tag{14}$$

c) The sequence of functions $J_k : \mathcal{Q} \to \mathbb{R}$, $k \in \mathbb{Z}_+$, defined recursively by

$$J_0 = 0$$

$$J_{k+1} = \max\left\{0, \mathbb{T}(J_k)\right\} \tag{15}$$

converges. □

Here, cost function $\sigma(q, u, w) = \tau \mu_\Delta(w) - R - \hat{v}(q, u) - \tau \gamma_o \rho_\Delta(u)$ includes two parameters: R and τ. Ideally, we seek to maximize $R > 0$ for which there exists $J : \mathcal{Q} \to \mathbb{R}_+$, $\tau > 0$, such that (13) holds. Since the optimal R cannot be directly computed, a numerical search is carried out: The range of values of τ for which (12) can be met for $R = 0$ is computed. This range is then sampled to compute the largest value of R at each sampling, with the largest of those being the (suboptimal) guaranteed rate of convergence.

V. ILLUSTRATIVE EXAMPLE

While not admitting a *quadratically* stabilizing controller, the deceptively simple example presented is nonetheless amenable to exact analysis in the full state feedback case, allowing us to numerically compare the performance of our finite state controllers under binary sensing limitations to the optimal performance in the ideal setting.

Indeed, when the switching controller has full access to the state and switching can occur at any time, it is always possible to stabilize a pair of harmonic oscillators

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = k_i x_1$$

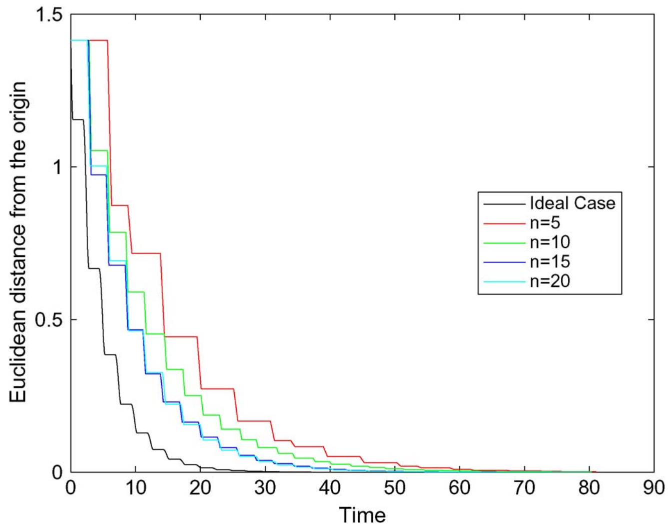

Fig. 8. Implementation of the DFM controller ($k_1 = -3$).

with $k_0 = -1$ and $k_1 \in \mathbb{R} \setminus \{-1\}$, by switching to k_1 exactly in those sectors where $sign(x_1 x_2) > 0$ if $k_1 < -1$ and where $sign(x_1 x_2) < 0$ if $k_1 > -1$. In contrast, when the plant is sampled and the sensor information available to the controller at the beginning of each sampling interval is restricted to the sign of the position measurement, the problem becomes much more difficult because of the state estimation problem (and the sampling, though this is not the motivation for our work). The approach described in this technical note will be used to design a stabilizing controller. The dynamics of the system, sampled at times tT_S, $t \in \mathbb{Z}_+$, are given by

$$x(t+1) = A_S(u(t))\,x(t)$$
$$y(t) = sgn(x_1(t))$$
$$v(t) = log\left(\frac{\|x(t+1)\|_2}{\|x(t)\|_2}\right)$$

where

$$A_S(u) = e^{A(u)T_S}, \quad A(u) = \begin{bmatrix} 0 & 1 \\ k_u & 0 \end{bmatrix}.$$

The unit circle is uniformly partitioned into $2n$ intervals and the sampling rate is matched to this partition by setting $T_S = \pi/n$, thus helping to counteract the conservatism introduced in quantifying Δ.

When $k_1 = -3$, $n = 5$ is the coarsest partition allowing for successful control design. Relevant data are shown in Table I (p is the number of iterations needed for the Value Iteration algorithm to converge), and representative implementations of the resulting closed loop systems as well as the optimal (full state feedback, unsampled) closed loop system are plotted in Fig. 8. Note that while R decreases for $n = 15$ and $n = 20$, the rate of convergence is still seen to improve, indicating that the actual rate of convergence may be significantly better than the provable rate. Beyond $n = 20$ the performance improvement tapers off, likely due to a combination of numerical errors and sampling effects.

VI. Conclusion

We demonstrated the use of a control oriented notion of finite state input/output approximation to systematically and efficiently synthesize certified-by-design stabilizing controllers for pairs of discrete-time, homogenous, unstable second order systems under binary sensor limitations. Future work will focus on reducing the conservatism of the approach, as well as demonstrating it for broader classes of plants and performance objectives.

Appendix

Four results are used in proving Theorem 4. The first (Theorem 5) is adapted from [33]; J is the storage function of a dissipative system with supply rate σ.

Theorem 5: Consider a DFM with state transition equation $q(t+1) = f(q(t), u(t), w(t))$ and let $\sigma : \mathcal{Q} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}$ be a given function. The following two statements are equivalent:

a) Inequality

$$\inf_{T \geq 0} \sum_{t=0}^{T} \sigma\left(q(t), u(t), w(t)\right) > -\infty$$

holds for all $\mathbf{u} \in \mathcal{U}^{\mathbb{Z}_+}$, $\mathbf{w} \in \mathcal{W}^{\mathbb{Z}_+}$ and $q(0) \in \mathcal{Q}$.

b) There exists a $J : \mathcal{Q} \to \mathbb{R}_+$ such that the inequality

$$J\left(f(q, u, w)\right) - J(q) \leq \sigma(q, u, w)$$

holds for all $q \in \mathcal{Q}$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$. □

Lemma 1: $\mathbb{T} : \mathbb{R}^{\mathcal{Q}} \to \mathbb{R}^{\mathcal{Q}}$ defined in (14) is monotonic, that is $J_1 \leq J_2 \Rightarrow \mathbb{T}(J_1) \leq \mathbb{T}(J_2)$.

Proof: If $J_1(q) \leq J_2(q)$, then

$$-\sigma(q, u, w) + J_1\left(f(q, u, w)\right) \leq -\sigma(q, u, w) + J_2\left(f(q, u, w)\right)$$

for all $u \in \mathcal{U}$, $w \in \mathcal{W}$ and $q \in \mathcal{Q}$. It follows that:

$$\min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} \left\{\sigma(q, u, w) + J_1\left(f(q, u, w)\right)\right\} \leq \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} \left\{\sigma(q, u, w) + J_2\left(f(q, u, w)\right)\right\}$$

for all $q \in \mathcal{Q}$, and hence $\mathbb{T}(J_1)(q) \leq \mathbb{T}(J_2)(q)$. ■

Lemma 2: Sequence $\{J_k\}$ defined in (15) is monotonically increasing.

Proof: The proof is by induction on k. We have $J_1 = \max\{0, \mathbb{T}(J_0)\} \geq 0 = J_0$. Now suppose $J_k \geq J_{k-1}$. Then $J_{k+1} = \max\{0, \mathbb{T}(J_k)\} \geq \max\{0, \mathbb{T}(J_{k-1})\} \geq J_k$. ■

Given $c \in \mathbb{R}$, $J : \mathcal{Q} \to \mathbb{R}_+$, consider function $J + c : \mathcal{Q} \to \mathbb{R}$ defined by $(J + c)(q) = J(q) + c$.

Lemma 3: For any $J : \mathcal{Q} \to \mathbb{R}_+$ and $c \in \mathbb{R}$, $\mathbb{T}(J + c) = \mathbb{T}(J) + c$.

Proof: For any $q \in \mathcal{Q}$ we have

$$\begin{aligned}\mathbb{T}\left((J+c)(q)\right) &= \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} \left\{\sigma(q, u, w) + (J+c)\left(f(q, u, w)\right)\right\} \\ &= \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} \left\{\sigma(q, u, w) + J\left(f(q, u, w)\right) + c\right\} \\ &= \mathbb{T}\left(J(q)\right) + c.\end{aligned}$$

■

Proof of Theorem 4: (b) ⇒ (a): Suppose there exists a J satisfying (13), and let

$$\varphi(q) = \arg\min_{u\in\mathcal{U}} \left\{\max_{w\in\mathcal{W}} \left(-\sigma(q, u, w) + J\left(f(q, u, w)\right)\right)\right\}.$$

For any $q \in \mathcal{Q}$, we have

$$\begin{aligned}J(q) &\geq \mathbb{T}\left(J(q)\right) \\ &= \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} \left\{-\sigma(q, u, w) + J\left(f(q, u, w)\right)\right\} \\ &= \max_{w\in\mathcal{W}} \left\{-\sigma\left(q, \varphi(q), w\right) + J\left(f\left(q, \varphi(q), w\right)\right)\right\} \\ &\geq -\sigma\left(q, \varphi(q), w\right) + J\left(f\left(q, \varphi(q), w\right)\right), \quad \forall w \in \mathcal{W}.\end{aligned}$$

It follows from Theorem 5 that (M, φ) satisfies (12).

(a) ⇒ (c): Suppose there exists a φ such that the closed loop system satisfies (12). By Theorem 5, there exists a $J : \mathcal{Q} \to \mathbb{R}_+$ such that

$J(f(q,\varphi(q),w)) - J(q) \le \sigma(q,\varphi(q),w), \forall q \in \mathcal{Q}, w \in \mathcal{W}$. For any $q \in \mathcal{Q}$, we have

$$\begin{aligned} J(q) &\ge -\sigma(q,\varphi(q),w) + J(f(q,\varphi(q),w)), \quad \forall w \in \mathcal{W} \\ &\ge \max_{w\in\mathcal{W}} \{-\sigma(q,\varphi(q),w) + J(f(q,\varphi(q),w))\} \\ &\ge \min_{u\in\mathcal{U}} \max_{w\in\mathcal{W}} \{-\sigma(q,u,w) + J(f(q,u,w))\} \\ &= \mathbb{T}(J(q)). \end{aligned}$$

It follows that $J+c$ also satisfies $(J+c) \ge \mathbb{T}((J+c))$ for any choice of $c \in \mathbb{R}$. Since $\mathcal{Q}$ is finite, we can assume, without loss of generality, that $J \ge 0$ with $\min_q J(q) = 0$ and $\max_q J(q) = m \ge 0$. Moreover, sequence $\{J_k\}$ defined in (15) is bounded above by J. The proof is by induction on k. We have $J_0 = 0 \le J$. Suppose that $J_k \le J$: Then $\mathbb{T}(J_k) \le \mathbb{T}(J) \le J$ where the first inequality follows from Lemma 1 and $J_{k+1} = \max\{0, \mathbb{T}(J_k)\} \le J$. Sequence $\{J_k\}$ is thus monotonically increasing (Lemma 2) and bounded above by J. Hence it converges to $J^* = \lim_{k\to\infty} J_k \le J$.

(c) ⇒ (b): Suppose that $\{J_k\}$ converges to $J^* = \lim_{k\to\infty} J_k$ and let $\epsilon_k = \max_{q\in\mathcal{Q}}\{J^*(q) - J_k(q)\}$. Note that $\epsilon_k \ge 0$, $\{\epsilon_k\}$ is monotonically decreasing (Lemma 2), and $\lim_{k\to\infty} \epsilon_k = 0$. Moreover $J_k(q) \ge J^*(q) - \epsilon_k, \forall q \in \mathcal{Q}$. It follows that $\mathbb{T}(J_k) \ge \mathbb{T}(J^* - \epsilon_k) = \mathbb{T}(J^*) - \epsilon_k$ (Lemmas 1, 3). But $J_{k+1} = \max\{0, \mathbb{T}(J_k)\} \ge \mathbb{T}(J^*) - \epsilon_k$. Thus $J^* = \lim_{k\to\infty} J_{k+1} \ge \mathbb{T}(J^*) - \lim_{k\to\infty} \epsilon_k = \mathbb{T}(J^*)$.

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