Value Iteration for (Switched) Homogeneous Systems

Michael Rinehart, Munther Dahleh, *Fellow, IEEE*, and Ilya Kolmanovsky, *Fellow, IEEE*

***Abstract*—In this note, we prove that dynamic programming value iteration converges uniformly for discrete-time homogeneous systems and continuous-time switched homogeneous systems. For discrete-time homogeneous systems, rather than discounting the cost function (which exponentially decreases the weights of the cost of future actions), we show that such systems satisfy approximate dynamic programming conditions recently developed by Rantzer, which provides a uniform bound on the convergence rate of value iteration over a compact set. For continuous-time switched homogeneous system, we present a transformation that generates an equivalent discrete-time homogeneous system with an additional "sampling" input for which discrete-time value iteration is compatible, and we further show that the inclusion of homogeneous switching costs results in a continuous value function.**

***Index Terms*—Dynamic programming, homogeneous systems, optimal control, switched systems.**

I. Introduction

In this technical note, we present new dynamic programming results for discrete-time homogeneous systems and continuous-time switched homogeneous systems. In particular, we provide conditions under which the value iteration algorithm [1] converges and the value function is continuous. Such convergence and continuity results may be used to compute approximately-optimal control laws for these systems. The problem formulation covers, as special cases, switched linear systems and nonlinear switched systems for which accurate homogeneous approximations can be developed.

Under fairly general conditions, value iteration is guaranteed to converge, but not necessarily to the value function. For infinite horizon formulations, a discounted cost function [1] in the Bellman equation may be used to guarantee the convergence of the value iteration algorithm, but at the price of changing the desired performance of the system. In [2], a sufficient condition on the value function is presented that guarantees the convergence of value iteration. In particular, it is shown that if the value function is uniformly bounded by a fixed proportion of the incremental cost function, then value iteration converges uniformly.

In this technical note, we prove that, under mild conditions, discrete-time homogeneous systems and continuous-time switched homogeneous systems satisfy the conditions in [2] for the uniform convergence of value iteration. Furthermore, the continuous-time value function is shown to be continuous. In the case of continuous-time systems, we present a method for transforming the system into a discrete-time system with an additional "sampling" input that makes it compatible with value iteration. As an application, we derive some of the results related to the work of Tuna in [3] but specialized for the optimal control of switched homogeneous systems. The results of this technical note provide an alternate framework for stability and performance analysis of homogeneous systems by leveraging the properties of such systems in a new dynamic programming framework that can simplify such analysis.

Manuscript received September 15, 2008; revised December 20, 2008. First published May 27, 2009; current version published June 10, 2009. This work was supported by the Ford Research and Advanced Engineering, Ford Motor Company, Dearborn, MI. Recommended by Associate Editor D. Liberzon.

M. Rinehart and M. Dahleh are with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02142 USA (e-mail: mdrine@mit.edu; dahleh@mit.edu).

I. Kolmanovsky is with Ford Research and Advanced Engineering, Ford Motor Company, Dearborn, MI 48126 USA (e-mail: ikolmano@ford.com).

Digital Object Identifier 10.1109/TAC.2009.2013055

II. Discrete-Time and Continuous-Time Homogeneous Systems

A. Notation and Assumptions for Discrete-Time and Continuous-Time Homogeneous Systems

We first begin with the definition of homogeneous systems used in this technical note.

Definition 1: A function $h : Y \times U \rightarrow H$ is **degree-**d **homogeneous-in-**y if there exists a matrix function $G(\alpha) = diag(\alpha^{r_1}, \ldots, \alpha^{r_n})$ for positive real constants α and r_i such that

$$h\left(G(\alpha)y, u\right) = \alpha^{d-1} G(\alpha) h(y, u).$$

For ease, we will only consider the case $G(\alpha) = \alpha$, and the results of the technical note may easily be rewritten for general G (in the switched-system case, the systems must share the same G). A consequence of this assumption is that we can restrict our analysis of the system to the unit sphere in $\Re^n$, which we denote as S^{n-1}.

In this technical note, we consider a discrete-time (DT) homogeneous system of the form

$$x(t+1) = f\left(x(t), u(t)\right) \tag{1}$$

where t is an integer, the state $x(t)$ is a vector in $\Re^n$, the input $u(t)$ is a vector in some compact set U, and f is degree-1 homogeneous-in-x.

Remark 1: As it is not generally desirable to apply unbounded u for bounded x, homogeneity in the parameter u in Definition 1 is not necessary [4]. If $\tilde{f}$ is degree-1 homogeneous in x and u, we can apply the transformation $f(x, u) = \tilde{f}(x, \|x\|u)$ and restrict u to some bounded set.

We also consider a continuous-time (CT) switched homogeneous system of the form

$$\dot{y}(\tau) = g_{i(\tau)}\left(y(\tau)\right) \tag{2}$$

where $\tau \in \Re$, the state $y(\tau)$ is a vector in $\Re^n$, the mode input $i(\tau)$ is a piecewise-constant function continuous from the right and taking values in a finite set Q (the set of *modes*), and the function g_i is a degree-d_i ($d_i \geq 1$) homogeneous-in-y function for each mode $i \in Q$.

We now define several important notations used throughout the technical note for CT switched systems:

- let $\tau_0 = 0$ and successively define the k^{th} *switching instance* τ_k as the first time $i(\tau)$ changes value since time τ_{k-1}, i.e. $\tau_k = \min\{\tau > \tau_{k-1} | i(\tau^-) \neq i(\tau)\}$
- define $y_k = y(\tau_k)$ as the k^{th} *switching state*,
- and define $i_k = i(\tau_k)$ as the k^{th} *operating mode* and denote the *mode sequence* as the list $(i_0, i_1, \ldots)$.

If the mode becomes a constant after some switching time t_k, i.e. $i(\tau) = a$ is constant for $\tau \geq \tau_k$, then as there are no more switches, we define $\tau_j = \infty$ and $i_j = a$ for all integers $j > k$. We also let $i_{-1} = i_0$, which will help simplify notation.

Finally, it will be useful to explicitly express the trajectory of (2) as a function of time, the initial condition, and the input i. Denote the value at time τ of the trajectory originating from y_0 under a switching law i

as $y(\tau, y_0, i)$ for those values of τ for which the trajectory is defined (since the trajectory may possess finite escape time).

We conclude this section with some assumptions about the CT and DT systems.

Assumption 1: The functions g_i are locally Lipschitz.

Assumption 2: f is bounded over $S^{n-1} \times U$.

Note that Assumption 2 automatically holds if f is continuous.

III. DT Bellman Equation and Value Iteration

For the DT system (1), let V be the *DT cost function* given by

$$V(x_0, u) = \sum_{t=0}^{\infty} L\left(x(t), u(t)\right) \tag{3}$$

where L is positive-definite and degree-d, $d > 1$, homogeneous in x. Define the *DT value function* as

$$V^*(x) = \inf_{u \in U} V(x, u).$$

By the homogeneity of L, it is clear that V^* is also degree-d homogeneous.

It is well known that the value function satisfies the Bellman equation

$$V^*(x) = \inf_{u \in U} \left\{V^*\left(f(x,u)\right) + L(x,u)\right\}. \tag{4}$$

If the value function V^* is known, the optimal policy can be computed through an evaluation of the expression

$$u^*(x) \in \arg\min_{u \in U} \left\{V^*\left(f(x,u)\right) + L(x,u)\right\}$$

if the minimum exists.

A means for approximating the value function is by *value iteration*, where successively-improving approximations to the value function are computed iteratively in the following manner: pick some function V^0 on $\Re^n$ and compute the sequence $(V^1, V^2, \ldots)$ iteratively by the relation

$$V^{k+1}(x) = \inf_{u \in U} \left\{V^k\left(f(x,u)\right) + L(x,u)\right\}. \tag{5}$$

If the limit exists, denote $V^\infty = \lim_{k\to\infty} V^k$.

A. Convergence of DT Value Iteration and Continuity of the Value Function

While it is not generally true that value iteration will converge to the value function, certain assumptions may be imposed to guarantee convergence. In this technical note, we make use of a convergence result given in [2], which we restate here in a form more amenable to our framework.

Proposition 1: If $V^*(f(x,u)) \leq \gamma L(x,u)$ holds uniformly for some constant $\gamma \geq 0$ and if V^* is bounded over a compact set E, then $(V^k)_k$[1] converges uniformly to V^* over E.

Proof: According to [2], for $\alpha V^* \leq V^0 \leq \beta V^*$

$$\left[1 + \frac{\alpha^{-1} - 1}{(1+\gamma^{-1})^k}\right]^{-1} V^*(x) \leq V^k(x) \leq \left[1 + \frac{\beta - 1}{(1+\gamma^{-1})^k}\right] V^*(x).$$

[1]We use the notation $(\cdot)_k$ to indicate a sequence over the index k, which will be useful in later sections when additional subscripts may be present in the sequences.

Uniform convergence is a consequence of the fact that V^* is bounded over E. ■

The results of this technical note result in part by showing that homogeneous systems satisfy the conditions of Proposition 1. We now state an immediate corollary of this result.

Corollary 1: If $L(S^{n-1}, U)$ is lower bounded by a positive constant and $V^*(S^{n-1})$ is bounded, then $(V^k)_k$ converges uniformly to V^* over S^{n-1}

Proof: By homogeneity, V^* is bounded over any compact set in $\Re^n$, in particular the compact set containing $f(S^{n-1}, U)$. Therefore, there exists a $\gamma > 0$ such that $V^* \circ f(x,u) < \gamma L(x,u)$ for all $(x,u) \in S^{n-1} \times U$. By homogeneity, the inequality extends over $\Re^n \times U$, and so uniform convergence results from Proposition 1 with $E = S^{n-1}$. ■

We now state a corollary concerning the continuity of the DT value function.

Corollary 2: If $L(S^{n-1}, U)$ is lower bounded by a positive constant, $V^*(S^{n-1})$ is bounded, and V^k is continuous for all k, then $V^\infty = V^*$ and V^* is continuous.

Proof: By Corollary 1, value iteration is uniformly convergent. Since V^k is continuous for all k, V^* is continuous over S^{n-1} and, by homogeneity, continuous over $\Re^n$ as well. ■

Finally, it may be of interest to determine the boundedness of V^* from value iteration, and we state a useful result concerning this test.

Proposition 2: If $V^0 = 0$, (5) is minimized by some u_k^* for each k, and $L(S^{n-1}, U)$ is lower bounded by a positive constant, then if $V^\infty(S^{n-1})$ is bounded, $V^*(S^{n-1})$ is bounded as well.

Proof: First, if $V^0 = 0$, then it can be shown that the sequence $(V^k(x))_k$ is monotonically increasing and bounded by $V^*(x)$.

Now, if the optimal input

$$u_k^*(x) \in \arg\min_u \left\{V^k\left(f(x,u)\right) + L(x,u)\right\}$$

exists, then we let

$$V_i^K(x) = \sum_{t=0}^{K-1} L\left(x(t), u_{K-t}^*\left(x(t)\right)\right).$$

We term $(u_K^*, u_{K-1}^*, \ldots, u_1^*)$ the *K-step roll-out policy* [1].

Choose $\alpha < 1$. Let $\lambda > 0$ be such that $L(S^{n-1}, U) > \lambda$. By homogeneity and by our assumptions, $L(x,u) > \|x\|^d \lambda$ for all x and u (note that d is the degree-of-homogeneity of L).

By the boundedness of $V^\infty(S^{n-1})$, there exists an integer K such that $V^\infty(S^{n-1}) < K\lambda\alpha^d$, which, since $V_i^K(x) \leq V^\infty(x)$, yields $V_i(x_0) < K\lambda\alpha^d \|x_0\|^d$.

Therefore, letting $x(t)$ result from an application of u_k^*, we have

$$\begin{aligned}\lambda \sum_{l=0}^{K-1} \|x(t)\|^d &\leq \sum_{t=0}^{K-1} L\left(x(t), u_{K-t}^*\left(x(t)\right)\right)\\ &= V_i^K(x_0)\\ &< K\lambda\alpha^d \|x(0)\|^d\end{aligned}$$

$$\text{Thus,} \ \sum_{l=0}^{K-1} \|x(t)\|^d < K\alpha^d \|x(0)\|^d.$$

Therefore, for some time $t(x_0) < K$, $\|x(t(x_0))\|^d < \alpha^d \|x(0)\|^d$. By repeated application of the K-step roll-out policy, it can be shown that the resulting cost can be bounded over $S^n - 1$ (the cost can be bounded by a geometric series since $d > 1$). Therefore, the optimal cost is bounded over S^{n-1} as well. ■

IV. CT Value Function

For the CT system (2), consider the *CT cost function* $J(y_0, i)$ for an input trajectory i defined as

$$J(y_0, i) = \int_0^\infty \|y(\tau, y_0, i(\tau))\|^{1+d_{i(\tau)}} d\tau + \sum_{k=0}^\infty \|y(\tau_k, y_0, i)\|^2 K_{i(\tau_k^-) i(\tau_k)} \quad (6)$$

where the switching-cost constants K_{mn} are nonnegative for $m \neq n$ and zero otherwise.

Optimizing over all switching laws i with initial mode i_0, we obtain the *CT value function*

$$J_{i_0}^*(y_0) = \inf_{\{i | i(0) = i_0\}} J(y_0, i). \quad (7)$$

A. Degree-1 Transformation of the CT System

To simplify the proofs of this section, we apply a useful transformation that will generate a degree-1 system having the same trajectories as the CT system (2). As in [5], let

$$\dot{z}(\tau) = \tilde{g}_{i(\tau)}(z(\tau)) = \|z(\tau)\|^{-d_{i(\tau)}+1} g_{i(\tau)}(z(\tau)). \quad (8)$$

Under suitable choices for each switching law,[2] both (2) and (8) generate the same trajectories, but (8) is degree-1 homogeneous by this rate transformation of (2).

Define a new cost function $\tilde{J}$ for system (8) as

$$\tilde{J}(z_0, i) = \int_0^\infty \|z(\tau, z_0, i)\|^2 d\tau + \sum_{k=0}^\infty \|z_k\|^2 K_{i_{k-1} i_k} \quad (9)$$

and define $\tilde{J}_{i_0}^*(y_0) = \inf_{\{i | i(0) = i_0\}} \tilde{J}(y_0, i)$. It is clear that $\tilde{J}_i^*$ is degree-2 homogeneous. We now state the useful consequence of this transformation, the proof of which can be found in [6].

Proposition 3: $J_i^* = \tilde{J}_i^*$.

B. Continuity of the CT Value Function

In the case of the CT system being asymptotically controllable, it is of interest to prove that the value function is continuous. To this end, we impose the following assumption on the system:

Assumption 3: The CT system (2) is asymptotically controllable [7], and there exists such a stabilizing control law that has a finite number of switches in any finite time interval.

To prove that J_i^* is continuous, we seek to leverage Corollary 2, but this result only applies to DT systems.[3] We now present a transformation of the CT to a DT system that will allow us to apply the DT value iteration results. First, we define a new function h_i representing the sampled dynamics of the normalized CT system (8) for a "sampling period" τ

$$h_i(x, \tau) = z(\tau, x, i).$$

We also define a new incremental cost function l as a sampling of the normalized cost (9)

$$l(x, \tau, i, j) = \int_0^\tau \|z(\gamma, x, i)\|^2 d\gamma + \|z(\tau, x, i)\|^2 K_{ij}.$$

If we treat i and τ as control inputs, we have a DT system

$$x(t+1) = h_{i(t)}(x(t), \tau(t)) \quad (10)$$

where the time t is a nonnegative integer. By substitution and by optimality, we can express $\tilde{J}_i^*$ by [4]

$$\tilde{J}_i^*(x) = \inf_{\{j, 0 \le \tau \le T_0\}} \left\{ \tilde{J}_j^*(h_i(x, \tau)) + l(x, \tau, i, j) \right\} \quad (11)$$

for any $T_0 > 0$. In essence, all we have done is split-up the expression of the value function by the switching times, which is possible by optimality. Also, by allowing "switches" to the current mode, we are able to restrict τ to a compact set.

We can now use value iteration to prove that J_i^* is continuous. Define the sequence $(\tilde{J}_i^k)_k$ by

$$\tilde{J}_i^{k+1}(x) = \inf_{\{j, 0 \le \tau \le T_0\}} \left\{ \tilde{J}_j^k(h_i(x, \tau)) + l(x, \tau, i, j) \right\}. \quad (12)$$

We first prove that value iteration converges for the CT system.

Proposition 4: If $K_{ij} > 0$ for $i \neq j$, then $(\tilde{J}_i^k)_k$ converges uniformly over S^{n-1} to $\tilde{J}_i^*$.

Proof: Define a new incremental cost $\hat{l}$ as

$$\hat{l}(x, \tau, i, j) = \begin{cases} \infty, & i = j \text{ and } \tau < T_0 \\ l(x, \tau, i, j) & \text{otherwise.} \end{cases}$$

The Bellman equation (11) may be equivalently written using $\hat{l}$ instead of l.

Let $I = [0, T_0]$. By Proposition A.1 (see Appendix), $\tilde{J}_i^*$ is bounded over any compact set, and therefore $\tilde{J}_i^* \circ h_i(S^{n-1}, I)$ is bounded for all i.

Since $\|z(I, S^{n-1}, Q)\|^2$ is a compact set not containing zero, it is lower bounded by a positive constant. Therefore, $\hat{l}(S^{n-1}, I, i, j)$ for $i \neq j$ and $\hat{l}(S^{n-1}, T_0, i, i)$ are lower bounded by a positive constant. For $\tau < T_0$, $\hat{l}(x, \tau, i, i) = \infty$, so it is trivially lower bounded.

Hence, there exists a positive constant γ such that $\tilde{J}_j^* \circ h_i \le \gamma \hat{l}(\cdot, \cdot, i, j)$ for all i, j over $S^{n-1} \times I$.

The boundedness condition of Proposition 1 (and hence uniform convergence over S^{n-1}) follows. ■

We now prove that the value function is continuous.

Theorem 1: If $K_{ij} > 0$ for $i \neq j$, then J_i^* is continuous.

Proof: We will construct a value iteration sequence to prove the claim. If we use Corollary 2, we need only to show that each $\tilde{J}_i^k$ of such a sequence is continuous. We proceed by induction.

Let $I = [0, T_0]$. First, define sets T_m satisfying 1) T_m is finite, 2) $T_m \subset T_{m+1}$,and 3) for all $\tau \in I$, there exists a $\hat{\tau} \in T_m$ such that $|\tau - \hat{\tau}| < (1/m)$. Basically, we are quantizing the values for τ.

Assume $\tilde{J}_j^k$ is continuous for all j. By continuity over the compact controller set $Q \times I$, the minimizers τ^* and j^* of (11) exist. Define $\tilde{J}_i^{k+1,m}$ by

$$\tilde{J}_i^{k+1,m}(x) = \min_{\{j, \tau \in T_m\}} \left\{ \tilde{J}_j^k(h_i(x, \tau)) + l(x, \tau, i, j) \right\}.$$

[2]The switching laws need to be scaled in time in order for the switchings to occur at the same location in the state space (i.e., so that $z_k = y_k$).

[3]Clearly, the results apply to DT switched systems by extending the input set U to $U \times Q$ in order to include the mode input.

[4]We note that i is actually a state of the DT system, but, for clarity, we write the value function using the index i_0 as in $J_{i_0^*}(x_0)$ instead of writing $J^*(x_0, i_0)$.

Clearly, $\tilde{J}_i^{k+1,m}(x) \geq \tilde{J}_i^{k+1}(x)$. Since $\tilde{J}_i^{k+1,m}$ is the minimum over a finite set of continuous functions, it is continuous.

Choose any $\epsilon > 0$. By the uniform continuities of $\tilde{J}_j^k \circ h_i$ and l over $S^{n-1} \times I$, there exists a δ such that

$$\left|\left(\tilde{J}_j^k\left(h_i(x,\tau)\right) + l(x,\tau,i,j)\right) - \left(\tilde{J}_j^k\left(h_i(x,\hat{\tau})\right) + l(x,\hat{\tau},i,j)\right)\right| < \epsilon$$

for $|\tau - \hat{\tau}| < \delta$ and for all $x \in S^{n-1}, i, j$.

Therefore, for all $x \in S^{n-1}$, there exists an M such that for all $m > M$

$$\begin{aligned}\left|\tilde{J}_i^{k+1,m}(x) - \tilde{J}_i^{k+1}(x)\right| &= \min_{\{j,\tau\in T_m\}}\left\{\tilde{J}_j^k\left(h_i(x,\tau)\right) + l(x,\tau,i,j)\right\} \\ &\quad - \left(\tilde{J}_{j^*}^k\left(h_i(x,\tau^*)\right) + l(x,\tau^*,i,j^*)\right) \\ &\leq \tilde{J}_{j^*}^k\left(h_i\left(x,\tau_m^*\right)\right) + l\left(x,\tau_m^*,i,j^*\right) \\ &\quad - \left(\tilde{J}_{j^*}^k\left(h_i(x,\tau^*)\right) + l(x,\tau^*,i,j^*)\right) \\ &\leq \epsilon\end{aligned}$$

where $\tau_m^* = \arg\min_{\tau\in T_m}|\tau - \tau^*|$. Consequently, $(\tilde{J}_i^{k+1,m})_m$ converges uniformly to $\tilde{J}_i^{k+1}$ over S^{n-1} and, hence, $\tilde{J}_i^{k+1}$ is continuous over S^{n-1}.

If we let $\tilde{J}_i^0 = 0$ (which is continuous), then by induction, $\tilde{J}_i^k$ is continuous for all k. Hence, $\tilde{J}_i^*$ is continuous. ■

V. Application to the Control of Switched Homogeneous Systems

In this section, we briefly apply the previous results to the control of CT switched homogeneous systems. In [3], it is shown that a state-dependent sampling time can be used to transform a CT homogeneous system into a degree-1 homogeneous DT system, and a feedback control law can be approximated using a quantization of the unit sphere. Therefore, we can assume, without loss of generality, that we are sampling the degree-1 CT system (8), for which we can apply a fixed sampling period T_0. In this section, we show that our techniques allow us to use simple inductive proofs to show the CT value function can be approximated and controlled in DT.

Define the DT incremental cost function as $L(x,i,j) = T_0\|x\|^2 + \|x\|^2 K_{ij}$, which serves as an approximation to $l(x,T_0,i,j)$ for small T_0. We now present two background results, the proofs of which are straight-forward and given in [6].

Proposition 5: For any $\epsilon > 0$, there exists a positive $\overline{T}_0$ such that $|l(x,\tau,i,j) - L(x,i,j)| < \epsilon$ for all $x \in S^{n-1}$, for all $0 \leq \tau \leq \overline{T}_0$, and for all i, j.

Proposition 6: For any $\epsilon > 0$, there exists a positive $\overline{T}_0$ such that $|\tilde{J}_j^*(h_i(x,\tau_1)) - \tilde{J}_j^*(h_i(x,\tau_2))| < \epsilon$ for all $x \in S^{n-1}$, for all $0 \leq \tau_1, \tau_2 \leq \overline{T}_0$, and for all i, j.

We now state the main result of this section.

Theorem 2 Approximation of the CT Value Function: For any $\epsilon > 0$, there exists a positive time $\overline{T}_0$ such that for all base sampling periods $T_0 \leq \overline{T}_0$, $|J_i^* - V_i^*| < \epsilon$ over S^{n-1}.

Proof: Let $\lambda > 0$ be such that $\|f_i(x)\|^2 < \lambda L(x,i,j)$ for all i, j and $x \in S^{n-1}$. If $\epsilon \geq (1/2\lambda)$, make it smaller, and choose $T_0 < \overline{T}_0$ for $\overline{T}_0$ given by Propositions 5 and 6 for the choice of ϵ. We now construct a value iteration sequence to prove the claim. Let $V_i^0 = 0$ and assume that for all $x \in S^{n-1}$ and i

$$V_i^k(x) \leq \frac{\tilde{J}_i^*(x) + 2\epsilon\|x\|^2}{1 - 2\lambda\epsilon}$$

An upper bound for V_i^{k+1} over S^{n-1} is

$$\begin{aligned}V_i^{k+1}(x) &= \min_j\left\{\|f_i(x)\|^2 V_j^k\left(\frac{f_i(x)}{\|f_i(x)\|}\right) + L(x,i,j)\right\} \\ &\leq \frac{\min_j\left\{\tilde{J}_j^*\left(f_i(x)\right) + 2\lambda\epsilon L(x,i,j) + (1-2\lambda\epsilon)L(x,i,j)\right\}}{1-2\lambda\epsilon} \\ &= \frac{\min_j\left\{\tilde{J}_j^*(f_i(x)) + L(x,i,j)\right\}}{1 - 2\lambda\epsilon}.\end{aligned}$$

Since

$$\begin{aligned}\tilde{J}_j^*\left(f_i(x)\right) = \tilde{J}_j^*\left(h_i(x,T_0)\right) &< \tilde{J}_j^*\left(h_i(x,\tau)\right) + \epsilon \\ L(x,i,j) &< l(x,\tau,i,j) + \epsilon\end{aligned}$$

for all $0 < \tau < T_0$, we have

$$\begin{aligned}V_i^{k+1}(x) &\leq \frac{\min_{\{j,0<\tau\leq T_0\}}\left\{\tilde{J}_j^*\left(h_i(x,\tau)\right) + l(x,\tau,i,j) + 2\epsilon\right\}}{1 - 2\lambda\epsilon} \\ &= \frac{\tilde{J}_i^*(x) + 2\epsilon}{1 - 2\lambda\epsilon}.\end{aligned}$$

A lower bound is similarly determined. Since $V_i^0 = 0$, induction holds, and by Proposition 2 and Corollary 1, value iteration converges. Because $\tilde{J}_i^*$ is upper and lower bounded over S^{n-1}, then, for sufficiently small ϵ, the approximation claim holds. ■

We now formally propose the existence of a stabilizing DT control law for the CT system, the full proof of which is given in [6].

Corollary 3 Stability of the CT System via DT Control: There exists a positive base sampling period T_0 such that the CT system (2) is asymptotically stable using the DT control law

$$u^*(x,i) \in \arg\min_j\left\{V_j^*\left(f_i(x)\right) + L(x,i,j)\right\}$$

Proof: The intuition behind the proof is that the CT trajectory can only deviate from an initial value on the unit sphere by a maximum distance in a sufficiently amount of time. At each time instance, the DT system's state is the initial value for the CT system, and hence there is a maximum deviation between the two over a time period. By homogeneity, this deviation attenuates proportionally as the DT system converges to the origin. ■

Remark 2: It is important to note that, in practice, the DT controller can only be semiglobally stabilizing since it is not possible to sample a CT system using arbitrarily short sampling periods as the state grows unbounded.

The reader is directed to [6] for additional results concerning a) the approximation of the approximating DT value function over the unit sphere, b) the construction of a DT controller for the CT system using a linear program, and c) proving the stability and approximate-optimality of the closed-loop system. All of these results are proven using simple inductive arguments based in the value iteration approach and results of this technical note.

VI. Conclusions

In this technical note, we presented conditions under which value iteration converges for discrete-time homogeneous and continuous-time switched homogeneous systems as well as conditions under which the value functions are continuous. Homogeneity was leveraged to show that the uniform convergence of value iteration results from the fact that such systems have value functions satisfying a boundedness condition presented in [2]. For continuous-time systems, a transformation of the system to a discrete-time homogeneous system was presented, and it was shown that the application of homogeneous switching costs guarantees the continuity of the value function. We applied these results and techniques to deriving simple proofs regarding the control of CT switched homogeneous systems.

Appendix
Background Results

Proposition A.1: $J_i^*(S^{n-1})$ is bounded.

Proof: Choose any $\epsilon < 1$. For each $z_0 \in S^{n-1}$, there exists a control law $i(z_0)$ and time $T(z_0)$ such that $\|z(\tau, z_0, i(z_0))\| < \epsilon$ for all $\tau \geq T(z_0)$. By continuity, there exists a distance $\delta(z_0)$ such that $\|z(T(z_0), \hat{z}_0, i(z_0))\| < \epsilon$ for all initial states $\|\hat{z}_0 - z_0\| < \delta(z_0)$. Choose M points $Z = \{z^1, z^2, \ldots, z^M\}$ on S^{n-1} such that the $\delta(z^k)$-neighborhoods about these points cover S^{n-1}. Let the function $\Gamma(z_0)$ map z_0 to its closest point in $\|z_0\|Z$ (basically, scale the quantization set Z).

Letting $K(i,t) = \arg\max_k\{\tau_k \leq t\}$, define the truncated cost at time t as

$$\tilde{J}^t(z_0, i) = \int_0^t \|z(\tau, z_0, i)\|^2 \, d\tau + \sum_{k=0}^{K(i,t)} \|z_k\|^2 K_{i_{k-1} i_k}$$

which is continuous over z_0 so long as the trajectory does not suffer finite-escape time. Define the quantized control law $\hat{i}(z_0) = i(\Gamma(z_0))$ and quantized time $\hat{T}(z_0) = T(\Gamma(z_0))$. We will bound the cost of control using

$$\tilde{J}_{\max} = \max_{z_0 \in S^{n-1}} \tilde{J}^{\hat{T}(z_0)}\left(z_0, \hat{i}(z_0)\right).$$

By homogeneity, $\|z(\hat{T}(z_0), z_0, \hat{i}(z_0))\| < \|z_0\|\epsilon$. Now since

$$\tilde{J}^{\hat{T}(z_0)}\left(z_0, \hat{i}(z_0)\right) \leq \|z_0\|^2 \tilde{J}_{\max}$$

we construct a stabilizing quasi-feedback control law as follows: execute $\hat{i}(z_0)$ until $\|z\| < \epsilon$, then execute $\hat{i}(z)$ until $\|z\| < \epsilon^2$, and so on. The cost of this non-optimal control law is bounded (by a geometric series). ■

Acknowledgment

The authors wish to thank Dr. D. Hrovat for his initiation of this joint project.

References

[1] D. Bertsekas, *Dynamic Programming and Optimal Control (vol. I)*, 3rd ed. Nashua, NH: Athena Scientific, 2005.
[2] A. Rantzer, "Relaxed dynamic programming in switching systems," *Proc. Inst. Elect. Eng.*, vol. 153, pp. 567–574, 2006.
[3] S. E. Tuna, "Optimal regulation of homogeneous systems," *Automatica*, vol. 41, no. 1, pp. 1879–1890, 2005.
[4] L. Rosier, "Homogeneous lyapunov function for homogeneous continuous vector field," *Syst. Control Lett.*, vol. 19, no. 6, pp. 467–473, Dec. 1992.
[5] L. Grune, "Homogeneous state feedback stabilization of homogeneous control systems," *SIAM J. Control Optim.*, vol. 38, pp. 1288–1314, 2000.
[6] M. Rinehart, M. Dahleh, and I. Kolmanovsky, "Value iteration and optimal control of (switched) homogeneous systems," LIDS Tech. Rep., 2009.
[7] F. Clarke, Y. Ledyaev, E. Sontag, and A. Subbotin, "Asymptotic controllability implies feedback stabilization," *IEEE Trans. Automat. Control*, vol. 45, no. 1, pp. 165–176, Jan. 1999.

Robust Stability and Stabilization of Fractional-Order Interval Systems: An LMI Approach

Jun-Guo Lu and Guanrong Chen

***Abstract*—This technical note presents necessary and sufficient conditions for the stability and stabilization of fractional-order interval systems. The results are obtained in terms of linear matrix inequalities. Two illustrative examples are given to show that our results are effective and less conservative for checking the robust stability and designing the stabilizing controller for fractional-order interval systems.**

***Index Terms*—Fractional-order system, interval system, linear matrix inequality (LMI), robust stability, robust stabilization.**

I. Introduction

Recently, fractional-order control systems have attracted increasing interest [1]–[6]. This is mainly due to the fact that many real-world physical systems are well characterized by fractional-order state equations [1], i.e., equations involving the so-called fractional derivatives and integrals. On the other hand, with the success in the synthesis of real noninteger differentiators and the emergence of a new electrical circuit element called "fractance" [7], [8], fractional-order controllers [9]–[12] have been designed and applied to control a variety of dynamical processes, including integer-order and fractional-order systems, so as to enhance the robustness and performance of the control systems.

Stability is fundamental to all control systems, certainly including fractional-order control systems. In [13]–[23], stability analyses on fractional-order control systems were presented. For interval fractional-order linear time-invariant (FO-LTI) systems, the stability issue was discussed first in [19] and then further in [20], even with fractional-order interval uncertainties. Note that, in [19], [20], the results were based on an experimentally verified Kharitonov-like procedure and only for SISO (single-input single-output) FO-LTI systems. For uncertain FO-LTI systems with interval coefficients described in the state-space form, the robust stability problem was tackled in [21], where the matrix perturbation theory was used to find the ranges of

Manuscript received July 31, 2008; revised November 27, 2008 and December 22, 2008. First published May 27, 2009; current version published June 10, 2009. This work was supported in part by the National Natural Science Foundation of China under Grants 60404005, 60744002, and 60775062, the National High Technology Research and Development Program of China under Grant 2006AA040203, the Program for New Century Excellent Talents in University under Grant NCET-07-0538, and the Hong Kong RGC-NSFC Joint Research Scheme under Grant N_CityU 107/07. Recommended by Associate Editor K. A. Morris.

J.-G. Lu is with the Department of Automation, Shanghai Jiaotong University, Shanghai 200240, China (e-mail:jglu@sjtu.edu.cn).

G. Chen is with the Department of Electronic Engineering, City University of Hong Kong, Kowloon, Hong Kong SAR, China (e-mail: eegchen@cityu.edu.hk).

Digital Object Identifier 10.1109/TAC.2009.2013056