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Stabilizability of two-dimensional linear systems via switched output feedback

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Abstract

The problem of stabilizing a second-order SISO LTI system of the form $\dot{x} = Ax + Bu$, $y = Cx$ with feedback of the form $u(x) = v(x)Cx$ is considered, where $v(x)$ is real-valued and has domain which is all of $\mathbb{R}^2$. It is shown that, when stabilization is possible, $v(x)$ can be chosen to take on no more than two values throughout the entire state-space (i.e., $v(x) \in \{k_1, k_2\}$ for all x and for some k_1, k_2), and an algorithm for finding a specific choice of $v(x)$ is presented. It is also shown that the classical root locus of the corresponding transfer function $C(sI - A)^{-1}B$ has a strong connection to this stabilization problem, and its utility is demonstrated through examples.

Keywords: Switched systems; Hybrid systems; Asymptotic stability; Root locus; Output feedback

1. Introduction

The study of hybrid systems is an area that has pervaded research for more than a decade (see, e.g., [2–6,8,9,11,13,14]). In particular, stabilization of continuous time systems via hybrid feedback is a problem that has received much attention in the recent literature. Artstein first addressed this question through examples [1]. Litsyn et al. show in [10] that the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

with (A, B) reachable and (C, A) observable can be stabilized via a hybrid feedback controller which uses a countable number of discrete states (and no continuous states) and which only depends upon the output y as opposed to the entire continuous state x. A natural question arises as to whether a hybrid feedback controller can be designed which uses a *finite* number of states instead. For the most part, the answer to this question is still open, though a partial answer has been given by Hu et al. in [7] based upon the so-called conic switching laws of [15,16]. In [7], it is shown that, for a certain class of single-input, single-output (SISO) second-order systems which are reachable and observable, there exists a feedback control law of the form $u(x) = v(x)Cx$ where

$$v(x) = \begin{cases} k_1 & \text{if } x_1x_2 \geqslant 0, \\ k_2 & \text{if } x_1x_2 < 0 \end{cases} \tag{2}$$

with $x = [x_1 \ \ x_2]'$ such that the resulting closed-loop system

$$\dot{x} = Ax + v(x)BCx \tag{3}$$

is globally exponentially stable. A control law of the form (2) is desirable as it can be implemented as a switch between two static gains which multiplies the output $y = Cx$. Note that, in general, the above strategy does not always work as the result of [10] sometimes requires a more complicated hybrid feedback structure to achieve stability, even when the system described by (1) is reachable and observable.

Example 1.1. Consider (1) with

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1].$$

This system is reachable and observable, but (3) is not stable for *any* real-valued choice of $v(x) \equiv v(x_1, x_2)$, not just $v(x_1, x_2)$ of

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the form (2). To see this, first note that the region $x_1 < 0, x_2 > 0$ is invariant under the flow of (3) for any choice of $v(x_1, x_2)$. Indeed, when $x_1 = 0$, $\dot{x}_1 = -x_2 < 0$, and when $x_2 = 0$, $\dot{x}_2 = -x_1 > 0$ for all choices of $v(x)$. Moreover, when $x_1(0) < 0$ and $x_2(0) > 0$, $\dot{x}_1 = 2x_1 - x_2 < 0$, which means that $x_1(t)$ is strictly decreasing, and, hence, does not decay to zero regardless of the choice of $v(x_1, x_2)$.

The goal of this paper is to answer the following questions: under what conditions on $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$ and $C \in \mathbb{R}^{1\times 2}$ can the closed-loop system (3) be made asymptotically stable for some choice of $v(x_1, x_2)$? And, moreover, when stability is achievable, how may one design $v(x_1, x_2)$ explicitly? As it turns out, the answer to the first question has a strong connection to the classical control notion of root locus. Essentially, if one considers control laws of the form $v(x_1, x_2)=k$ for some $k \in \mathbb{R}$, then the system (3) is stabilizable in only one of two situations:

- There exists a value of k such that the matrix $A + kBC$ is Hurwitz and, hence, (3) is exponentially stabilizable via static output feedback.
- There is no value of k for which $A + kBC$ is Hurwitz, but there does exist a value of k for which the eigenvalues of $A + kBC$ are complex. In this case, $v(x_1, x_2)$ can be chosen to take on only two values k_1 and k_2 throughout the entire state-space, i.e., $v(x_1, x_2) \in \{k_1, k_2\}$, where k_1 and k_2 are appropriately selected real constants, and global exponential stability can be achieved.

A third situation can exist in which there exists no value of k for which $A + kBC$ is Hurwitz and the eigenvalues of $A + kBC$ are real for all k. It is precisely these situations for which no choice of $v(x_1, x_2)$ will yield asymptotic stability.

Note that, unlike [10], the switching strategies employed here and in [7] in general require full knowledge of the state x of (1) rather than just knowing the output $y = Cx$. While we will not formally show this here, an appropriate first-order LTI observer of the plant state x can be designed to implement a slight variant of the control laws we discuss here (see [12] for details of this work). The work we discuss here is a necessary precursor to this more general problem, much like the linear system pole placement problem via state feedback is a precursor to the pole placement problem via output feedback.

The structure of the paper is as follows. First, we examine two particular case studies in which the form of the B and C vectors have special structure and analyze the conditions on the matrix A which will guarantee stability. Also, we will derive explicit forms for $v(x_1, x_2)$ which can be used to achieve stability when it is possible to do so. Next, we will show that, through appropriate coordinate transformations, all nontrivial[1] problems can be transformed into either one of these two case studies and then will use this to establish the main result. Finally, we explore a general method of designing such controllers (when they exist) and provide several examples to illustrate the methodology.

[1] By "nontrivial", we refer to problems in which neither B nor C is identically 0.

2. Case studies

In this section, we explore two specific case studies in which the A, B, and C matrices of (1) have particular structures. Using appropriate coordinate transformations, we will then relate the results of this section to derive the main result for general A, B, and C.

2.1. Case 1

We first assume a system of the following structure:

$$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1], \tag{4}$$

where $a, c \in \mathbb{R}$, and $b \geqslant 0$. Here, (3) takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & v(x_1, x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5}$$

We summarize the possibilities for stabilizability as a function of the parameters a, b, and c in the proposition below:

Proposition 2.1. *For system* (5):

(1) *If* $bc = 0$, *then* (5) *is exponentially stabilizable via static output feedback if* $a < 0$ *and is not stabilizable for any choice of* $v(x_1, x_2)$ *otherwise*.
(2) *If* $b > 0$ *and* $c > 0$, *when* $v(x_1, x_2) = k$ *for some constant* k, *then the eigenvalues of* (5) *are real for all* k, *and* (5) *is either exponentially stabilizable via static output feedback or is not stabilizable by any choice of* $v(x_1, x_2)$.
(3) *If* $b > 0$ *and* $c < 0$, *when* $v(x_1, x_2) = k$ *for some constant* k, *then the eigenvalues of* (5) *are not real for all* k, *and* (5) *is exponentially stabilizable either by static output feedback or by feedback of the form*

$$v(x_1, x_2) = \begin{cases} k_1 & \text{if } w_1' x = 0, \\ k_2 & \text{if } w_1' x \neq 0 \end{cases}$$

for some appropriate choice of w_1, k_1, *and* k_2.

We prove each part separately below.

Proof of Part 1. Note that if $b = 0$, the system described by (4) has an uncontrollable mode. In this case, stabilizability is possible if and only if $a < 0$ and can be achieved via $v(x_1, x_2) = k$, where $k < 0$. In a similar vein, if $c=0$, (4) has an unobservable mode. Noting that any initial condition with $x_2(0) = 0$ satisfies $x_2(t) = 0$ for all t, it is again clear that stabilizability is possible if and only if $a < 0$ and can be achieved by setting $v(x_1, x_2)$ to a negative real constant.

Proof of Part 2. If we set $v(x_1, x_2) = k$ for some constant k, the characteristic polynomial of (5) is given by

$$s^2 - (a + k)s + ak - bc. \tag{6}$$

First note that both roots of (6) are real for any value of k since the discriminant $(a+k)^2-4ak+4bc=(a-k)^2+4bc>0$ for all k. Now, both eigenvalues of (5) can be placed in the open left half plane if and only if there exists a value of k such that $a+k<0$ and $ak-bc>0$. When $a<0$, there always exists a value of k which satisfies both of these constraints and, hence, (5) is stabilizable via static output feedback.

When $a\geqslant 0$, there is no value k which can satisfy both inequalities simultaneously when $b>0$ and $c>0$. Hence, (5) cannot be stabilized via static output feedback. To show that (5) cannot be stabilized for *any* choice of $v(x_1,x_2)$, first recognize that, when $b>0$ and $c>0$, the conic region $x_1>0$, $x_2>0$ is invariant under the flow of (5) for any choice of $v(x_1,x_2)$. To show this, assume that the statement is not true, and that there exists a trajectory with $x_1(0)>0, x_2(0)>0$ that leaves the open first quadrant by crossing the axis $x_1=0$. At the point of time that the trajectory crosses the x_1 axis, the corresponding value of $\dot{x}_1$ is given by $bx_2>0$ which means that $x_1(t)$ must be *increasing* when it crosses the x_1 axis, an obvious contradiction. Similarly, if there exists some choice of $v(x_1,x_2)$ such that a trajectory escapes the open first quadrant by crossing the x_2 axis, at the time of crossing, $\dot{x}_2=cx_1>0$.

If $a\geqslant 0$, $b>0$, $c>0$, then (5) is not stabilizable for any choice of $v(x_1,x_2)$ for essentially the same reason as was presented in Example 1.1. By virtue of the above, if $x_1(0)>0, x_2(0)>0$, then $\dot{x}_1=ax_1+bx_2>0$, which means that $x_1(t)$ is always increasing for any choice of $v(x_1,x_2)$.

Proof of Part 3. When $c<0$, the roots of (6) can be made to lie in the open left half plane when $a<0$. When $a\geqslant 0$, the roots can also be made to lie in the open left half plane if and only if $a^2<-bc$. Hence, (5) is not static output feedback stabilizable if $a^2\geqslant -bc$, yet, as we now show, there exists a choice of $v(x_1,x_2)$ which yields global exponential stability. Closer examination of the characteristic polynomial (6) with $b>0$, $c<0$, and $a^2\geqslant -bc$ yields the following two statements:

- The roots of (6) are complex with nonnegative real part whenever $a-2\sqrt{-bc}<k<a+2\sqrt{-bc}$.
- There exists a negative real root of (6) whenever $k<-a$.

Since the roots of (6) can be calculated explicitly as

$$s=\frac{a+k}{2}\pm\frac{\sqrt{(a-k)^2+4bc}}{2}, \tag{7}$$

it is clear that the roots are complex whenever the first bulleted item holds. Moreover, the real part of the roots is nonnegative since, due to the fact that $a^2\geqslant -bc$,

$$\frac{a+k}{2}>a-\sqrt{-bc}\geqslant 0.$$

Now, when $k<-a$, the discriminant satisfies

$$(a-k)^2+4bc>4a^2+4bc\geqslant 0,$$

hence, the roots are real. Moreover, because the sum of the roots $(a+k)/2<0$, one root must be negative.

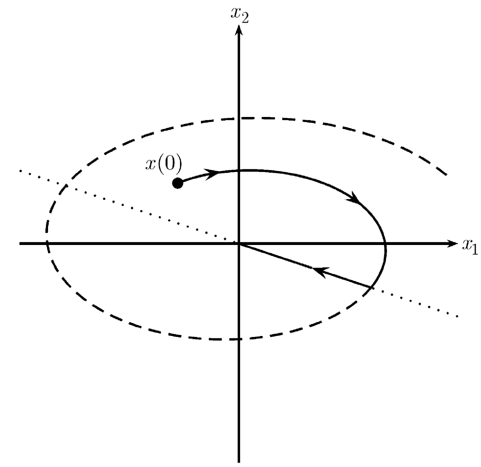

Fig. 1. Illustration of stabilization algorithm for a system which is not static output feedback stabilizable.

Informally speaking, to find a choice of $v(x_1,x_2)$ which asymptotically stabilizes (5), we use the following basic design strategy. The above analysis shows that there exists a value of k_1 which yields a real eigenvalue $\lambda_1<0$ and corresponding real eigenvector q_1. If we set $v(x_1,x_2)=k_1$ along q_1, then any initial condition which lies along q_1 will decay exponentially with rate λ_1. For all other values of x_1 and x_2 which do not lie along q_1, we find a value k_2 for which the eigenvalues are complex. If we set $v(x_1,x_2)=k_2$ everywhere else in the state-space, then any initial condition which does not lie along q_1 will rotate until it eventually "hits" q_1 and will decay exponentially thereafter. This idea is illustrated graphically in Fig. 1. Here, the dotted line represents the stable eigenvector q_1 when $v(x_1,x_2)=k_1$, the dashed line represents a sample phase portrait with initial condition $x(0)$ when $v(x_1,x_2)=k_2$ throughout the entire state-space, and the solid curve represents the trajectory with initial condition $x(0)$ when $v(x_1,x_2)=k_1$ along q_1 and $v(x_1,x_2)=k_2$ everywhere else in the state-space.

Before we prove this result formally, we need the following lemma:

Lemma 1. *Consider the linear system $\dot{z}=Az$ where $A\in\mathbb{R}^{2\times 2}$ has two complex conjugate eigenvalues. Then for any $w\in\mathbb{R}^2$ and any $z(0)$, there exists $t_0\in\mathbb{R}$ such that $w'z(t_0)=0$.*

Proof. If $w'z(0)=0$, then the statement immediately follows. Otherwise, without loss of generality, assume that $w'z(0)>0$. Because the eigenvalues of A are complex, the entries of the corresponding state transition matrix $\exp(At)$ are linear combinations of the terms $\exp(\sigma_0 t)\cos(\omega_0 t)$ and $\exp(\sigma_0 t)\sin(\omega_0 t)$ where $\omega_0>0$. Hence,

$$w'z\left(\frac{\pi}{\omega_0}\right)=-\exp\left(\frac{\sigma_0\pi}{\omega_0}\right)w'z(0)<0.$$

By continuity of $z(t)$, it then follows that there exists some time $t_0<\pi/\omega_0$ such that $w'z(t_0)=0$. $\square$

We now formally prove that the above informal description yields an exponentially stable system.

Proposition 2.2. *For system* (5) *with* $b>0$, $c<0$, *and* $a^2 \geqslant -bc$, *suppose that* k_1 *is chosen such that* (5) *with* $v(x_1,x_2)=k_1$ *has a stable eigenvector* q_1 *with corresponding eigenvalue* $\lambda_1<0$, *and* k_2 *is chosen such that* (5) *has two complex-conjugate eigenvalues. Let* w_1 *satisfy* $w_1'q_1=0$ *and consider*

$$v(x_1,x_2)=\begin{cases} k_1 & \text{if } w_1'x=0,\\ k_2 & \text{if } w_1'x\neq 0.\end{cases}$$

Then (5) *is globally exponentially stable for the above choice of* $v(x_1,x_2)$ *with decay rate* λ_1.

Proof. If $x(0)=\alpha q_1$ for some $\alpha\in\mathbb{R}$, then $x(t)=\exp(\lambda_1 t)x(0)$ and the statement holds. Otherwise, $w_1'x(0)\neq 0$, and, by virtue of Lemma 1, there exists some value of t_0 such that $w_1'x(t_0)=0$. In other words, $x(t_0)=\alpha q$ for some $\alpha\in\mathbb{R}$. Now, $x(t)=\exp(\lambda_1(t-t_0))x(t_0)$ for all $t>t_0$. □

2.2. Case 2

Now we assume a system of the following structure:

$$A=\begin{bmatrix} a & b\\ 0 & c\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix},\quad C=[1\quad 0], \tag{8}$$

where $a, c\in\mathbb{R}$, and $b\geqslant 0$. Here, (3) takes the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix} a & b\\ v(x_1,x_2) & c\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{9}$$

We summarize the possibilities for stabilizability as a function of the parameters a, b, and c in the proposition below:

Proposition 2.3. *For system* (9):

(1) *If* $b=0$, *then* (9) *is exponentially stabilizable via static output feedback if* $a<0$ *and* $c<0$ *and is not stabilizable for any choice of* $v(x_1,x_2)$ *otherwise.*
(2) *If* $b>0$, *when* $v(x_1,x_2)=k$ *for some constant* k, *the eigenvalues of* (9) *are not real for all* k, *and* (9) *is exponentially stabilizable either by static output feedback or by feedback of the form*

$$v(x_1,x_2)=\begin{cases} k_1 & \text{if } w_1'x=0,\\ k_2 & \text{if } w_1'x\neq 0\end{cases}$$

for some appropriate choice of w_1, k_1, *and* k_2.

We prove each part separately below.

Proof of Part 1. If $b=0$, the system described by (8) is both uncontrollable and unobservable. In this case, (9) is stabilizable if and only if $a<0$ and $c<0$. That (9) is unstable if $a\geqslant 0$ is clear; if $c\geqslant 0$, then any solution with initial condition $x_1(0)=0$ satisfies $\dot{x}_2=cx_2$ and, hence, (9) is unstable for any choice of $v(x_1,x_2)$.

Proof of Part 2. If we set $v(x_1,x_2)=k$ for some constant k, the characteristic polynomial of (9) is

$$s^2-(a+c)s+ac-bk. \tag{10}$$

It is clear that if $a+c<0$, then there always exists a choice of k such that $ac-bk>0$, and hence (9) can be stabilized via static output feedback. If $a+c\geqslant 0$, then (9) can be stabilized via a choice of $v(x_1,x_2)$ which takes on two values throughout the entire state-space in a manner similar to that of Case 1. A more detailed observation of the roots of (10) when $a+c\geqslant 0$ reveal the following two facts:

- The roots of (10) are complex with nonnegative real part whenever $k<-(a-c)^2/4b$.
- There exists a negative real root of (10) whenever $k>ac/b$.

Because the roots of (10) can be calculated explicitly as

$$s=\frac{a+c}{2}\pm\frac{\sqrt{(a-c)^2+4bk}}{2}, \tag{11}$$

it is clear that the roots are complex whenever the first bulleted item holds.

Now, if k is chosen such that a negative real root exists, then the inequality $a+c<\sqrt{(a-c)^2+4bk}$ must be satisfied. A simple calculation shows that this is equivalent to the second bulleted item.

Using this result, we can derive a stabilization algorithm which is completely analogous to the algorithm of the previous case:

Proposition 2.4. *For system* (9) *with* $b>0$ *and* $a+c\geqslant 0$, *suppose that* k_1 *is chosen such that* (9) *has a stable eigenvector* q_1 *with corresponding eigenvalue* $\lambda_1<0$, *and* k_2 *is chosen such that* (9) *has two complex eigenvalues. Let* w_1 *satisfy* $w_1'q_1=0$, *and consider*

$$v(x_1,x_2)=\begin{cases} k_1 & \text{if } w_1'x=0,\\ k_2 & \text{if } w_1'x\neq 0.\end{cases}$$

Then (5) *is globally exponentially stable for the above choice of* $v(x_1,x_2)$ *with decay rate* λ_1.

Proof. Same as the proof of Proposition 2.2. □

3. Main result

While the case studies of the prior section may seem constrained due to the very special structure of the A, B, and C matrices, an appropriate change of coordinates reveals that any second order system of the form (1) and can be transformed into either Case 1 or 2.

Lemma 2. *Consider matrices* $A\in\mathbb{R}^{2\times 2}$, $B\in\mathbb{R}^{2\times 1}$, *and* $C\in\mathbb{R}^{1\times 2}$ *where neither* B *nor* C *is identically* 0. *For any* *invertible matrix* $T \in \mathbb{R}^{2\times 2}$, *define the triplet* $(\tilde{A}, \tilde{B}, \tilde{C})$ *as* $(T^{-1}AT, T^{-1}B, CT)$, *and let*

$$\tilde{A} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the following statements hold:

(1) *If* $CB \neq 0$, *then* $\exists T$ *such that*

$$\tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{C} = [0 \quad \alpha]$$

with $\alpha \neq 0$ *and* $b \geqslant 0$.

(2) *If* $CB = 0$, *then* $\exists T$ *such that*

$$\tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{C} = [\alpha \quad 0]$$

with $\alpha \neq 0$ *and* $b \geqslant 0$.

Proof. Let $B = [\beta_1 \;\; \beta_2]'$, $C = [\gamma_1 \;\; \gamma_2]$. To prove the first result, direct computation shows that the matrix

$$T = \begin{bmatrix} \gamma_2 & \beta_1 \\ -\gamma_1 & \beta_2 \end{bmatrix}$$

is invertible since $\det(T) = \gamma_1\beta_1 + \gamma_2\beta_2 = CB \neq 0$. Moreover, $\tilde{B} = [0 \;\; 1]'$, $\tilde{C} = [0 \;\; \alpha]$ where $\alpha = CB \neq 0$. If $b \geqslant 0$, then the statement follows. Otherwise, the transformation

$$T_2 = T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\gamma_2 & \beta_1 \\ \gamma_1 & \beta_2 \end{bmatrix}$$

will satisfy all of the desired properties.

To prove the second part of the statement, consider the matrix

$$T = \begin{bmatrix} \beta_2 & \beta_1 \\ -\beta_1 & \beta_2 \end{bmatrix}.$$

Then $\det(T) = \beta_1^2 + \beta_2^2 \neq 0$, and, hence, T is invertible. Note that any nonzero C which satisfies $CB = 0$ may be written as $C = [\delta\beta_2 \;\; -\delta\beta_1]$, where $\delta \neq 0$. Hence, $\tilde{B} = [0 \;\; 1]'$, $\tilde{C} = [\alpha \;\; 0]$, where $\alpha = \delta(\beta_1^2 + \beta_2^2) \neq 0$. If $b \geqslant 0$, then the statement holds. Otherwise, the transformation

$$T_2 = T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\beta_2 & \beta_1 \\ \beta_1 & \beta_2 \end{bmatrix}$$

will satisfy all of the desired properties. □

We are now ready to present the main result of the paper.

Theorem 3. *Consider system* (1) *with* $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 1}$, *and* $C \in \mathbb{R}^{1\times 2}$ *where neither* C *nor* B *is identically* 0. *Define the root locus of this system to be the locus of eigenvalues of* (3) *when* $v(x_1, x_2) = k$ *as* k *varies continuously over* $\mathbb{R}$. *Then exactly one of the following statements is true*:

(1) *The system is static output feedback stabilizable*.
(2) *The system is not static output feedback stabilizable*, *but it has root locus which takes on complex values for some values of* $k \in \mathbb{R}$ *and is stabilizable by a control law* $v(x_1, x_2)$ *which takes on two values throughout the entire state-space*.
(3) *The system has a root locus which is real for all values of* $k \in \mathbb{R}$ *and is not stabilizable by control of the form* (3) *for any choice of* $v(x_1, x_2)$.

Proof. Using Lemma 2, whenever C and B are not identically 0, there exists a coordinate transformation where (3) is either of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d + \alpha v(x_1, x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c + \alpha v(x_1, x_2) & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

with $\alpha \neq 0$ and $b \geqslant 0$. Since $\alpha \neq 0$, the substitutions $\tilde{u}(x_1, x_2) = d + \alpha v(x_1, x_2)$ and $\tilde{u}(x_1, x_2) = c + \alpha v(x_1, x_2)$ are invertible. Hence, any system of the form (1) for which neither C nor B is identically 0 can be transformed into the form of either Case 1 or 2 of the previous section. Since the statements of the theorem were shown to be true for both of these case studies, it then follows that the result must hold in the more general setting. □

4. Connection to the classical root locus

Note that in order to obtain a stabilizing controller (when it exists), one need not carry out the transformations described in Lemma 2. Rather, one may analyze the root locus of the matrix $A + kBC$ directly and (when necessary) find a stable eigenvector to derive an appropriate control law $v(x_1, x_2)$. Moreover, when (A, B) is reachable and (C, A) is observable, we may employ classical root locus techniques to the corresponding transfer function $C(sI - A)^{-1}B$ to quickly ascertain the geometric behavior of the root locus. When either (A, B) is not reachable and/or (C, A) is not observable, we may still use classical root locus techniques on the transfer function $C(sI - A)^{-1}B$, but we must take care to include the unreachable and/or unobservable modes in our analysis. We now formalize these statements, beginning with the following lemma.

Lemma 4. *For* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, *and* $C \in \mathbb{R}^{1\times n}$, *if the pair* (A, B) *is reachable and the pair* (C, A) *is observable*, *then the eigenvalues of* $A + kBC$ *for each* $k \neq 0$ *are given by all values of* s *which satisfy* $1 - kC(sI - A)^{-1}B = 0$.

Proof. We first show that if $k \neq 0$, then no eigenvalue of A can be an eigenvalue of $A + kBC$. If we assume the contrary, that there exists an eigenvalue of A that is also an eigenvalue of $A + kBC$, then there exists a right eigenvector p such that $Ap = sp$ and $(A + kBC)p = sp$ for some $s \in \mathbb{C}$. Hence, $kBCp = 0$ which consequently implies that $Cp = 0$. Moreover, $CA^{n-1}p = CA^{n-2}p = \cdots = CAp = 0$, which implies that the observability matrix is not full rank, thereby contradicting the assumption of observability of the pair (C, A). By examining the left eigenvectors of A and $A+kBC$, one can similarly conclude that a common eigenvalue between these two matrices causes the matrix

$$[A^{n-1}B \;\; A^{n-2}B \;\; \cdots \;\; AB \;\; B]$$

to lose rank, thereby contradicting the assumption that the pair (A, B) is reachable.

The eigenvalues of $A+kBC$ are those values of s for which $\det(sI-A-kBC)=0$. From the above result, when $k \neq 0$, the matrix $sI-A$ must be invertible, and hence

$$\det(sI-A-kBC)=\det(sI-A)\det(I-k(sI-A)^{-1}BC).$$

Thus, eigenvalues of $A+kBC$ are those values of s for which $\det(I-k(sI-A)^{-1}BC)=0$. Using the identity $\det(I-AB)=\det(I-BA)$, we find that

$$\det(I-k(sI-A)^{-1}BC)=1-kC(sI-A)^{-1}B. \qquad \square$$

In cases where either (A, B) is unreachable and/or (C, A) is unobservable, we have the following corollary whose proof is immediate and is left to the reader:

Corollary 5. *For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$, the eigenvalues of $A+kBC$ for each $k \neq 0$ are given by $s \in \mathscr{M} \cup \mathscr{N} \cup \mathscr{O}$, where $\mathscr{M}$ is the set of unreachable modes of (A, B), $\mathscr{N}$ is the set of unobservable modes of (C, A), and $\mathscr{O}$ is the classical root locus of the transfer function $C(sI-A)^{-1}B$, i.e. $\mathscr{O}=\{s : 1-kC(sI-A)^{-1}B=0\}$.*

5. Design methodology and examples

Using the result of Corollary 5, we may employ the following basic algorithm to find a stabilizing controller when one exists:

(1) Compute the transfer function $C(sI-A)^{-1}B$ and examine the corresponding root locus of (1) (i.e. the roots of $1-kC(sI-A)^{-1}B$ as k varies over $\mathbb{R}$, along with any fixed unreachable and/or unobservable modes of the original state-space model).

(2) If examination of the root locus shows that there exists k_0 for which both of the eigenvalues of $A+k_0BC$ lie in the open left half-plane, find such a value of k_0 and choose $v(x_1, x_2)=k_0$ for all x.

(3) If examination of the root locus indicates that there exists a value k_1 for which one of the eigenvalues $A+k_1BC$ lies in the open left half-plane and a value k_2 for which the imaginary part of the eigenvalues is nonzero, find corresponding values of k_1 and k_2, along with the (real) eigenvector q_1 of $A+k_1BC$ corresponding to the stable eigenvalue. Choose $v(x_1, x_2)$ such that

$$v(x_1, x_2)=\begin{cases} k_1 & \text{if } w_1'x=0, \\ k_2 & \text{if } w_1'x \neq 0, \end{cases}$$

where w_1 satisfies $w_1'q_1=0$.

(4) If neither (2) nor (3) holds, declare the system unstabilizable by any choice of $v(x_1, x_2)$.

We now provide several examples to illustrate the general design methodology described here.

Example 5.2. We consider three reachable, observable systems of the form

$$\dot{x}=A_ix+B_iu, \quad y=C_ix, \quad i \in \{1, 2, 3\},$$

$$A_1=\begin{bmatrix} -6 & -6 \\ -6 & 7 \end{bmatrix}, \quad A_2=\begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}, \quad A_3=\begin{bmatrix} 0 & 1 \\ -12 & 7 \end{bmatrix},$$

$$B_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad B_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_3=\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1=[0 \;\; 1], \quad C_2=[1 \;\; 1], \quad C_3=[-2 \;\; 1].$$

The transfer functions $H_i(s)$ corresponding to each of these state-space descriptions are given by

$$H_1(s)=\frac{s}{s^2-s-6},$$
$$H_2(s)=\frac{s+1}{s^2-s-6},$$
$$H_3(s)=\frac{s-2}{s^2-7s+12}.$$

The root locus for each of the above transfer functions is depicted in Fig. 2. From the first root locus diagram for $H_1(s)$, it is clear that the root locus is real for all k, but the zero at $s=0$ prevents one eigenvalue from entering the left half plane. Hence, there is no switching control law of the form (2) which can asymptotically stabilize this system.

While the root locus for $H_2(s)$ is also real for all k, the presence of the zero at $s=-1$ allows both eigenvalues to lie in the open left half plane for sufficiently negative values of k. Indeed, when $k=-7$, the eigenvalues are approximately -5.83 and -0.17. Hence, the second system can be made stable via static output feedback.

The third system $H_3(s)$ has a root locus that takes on complex values for some negative values of k, but both eigenvalues never lie in the left half plane simultaneously. Nevertheless, *one* of the eigenvalues can be made negative for sufficiently negative values of k. Indeed, when $k=-20/3$, -1 is an eigenvalue of $A_3+kB_3C_3$ with corresponding eigenvector $q_1=[1 \;\; -1]'$. When $k=-1$, the eigenvalues of $A_3+kB_3C_3$ are complex $(3\pm \mathrm{i})$. Noting that $w_1=[1 \;\; 1]'$ satisfies $w_1'q_1=0$, a stabilizing switching controller is given by $u(x_1, x_2)=v(x_1, x_2)C_3x$, where $v(x_1, x_2)$ is given by

$$v(x_1, x_2)=\begin{cases} -\dfrac{20}{3} & \text{if } x_1+x_2=0, \\ -1 & \text{if } x_1+x_2 \neq 0. \end{cases}$$

Example 5.3. We now consider two unreachable systems of the form

$$\dot{x}=A_ix+Bu, \quad y=Cx,$$

where $B=[1 \;\; 0]$, $C=[1 \;\; 1]$ and

$$A_1=\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

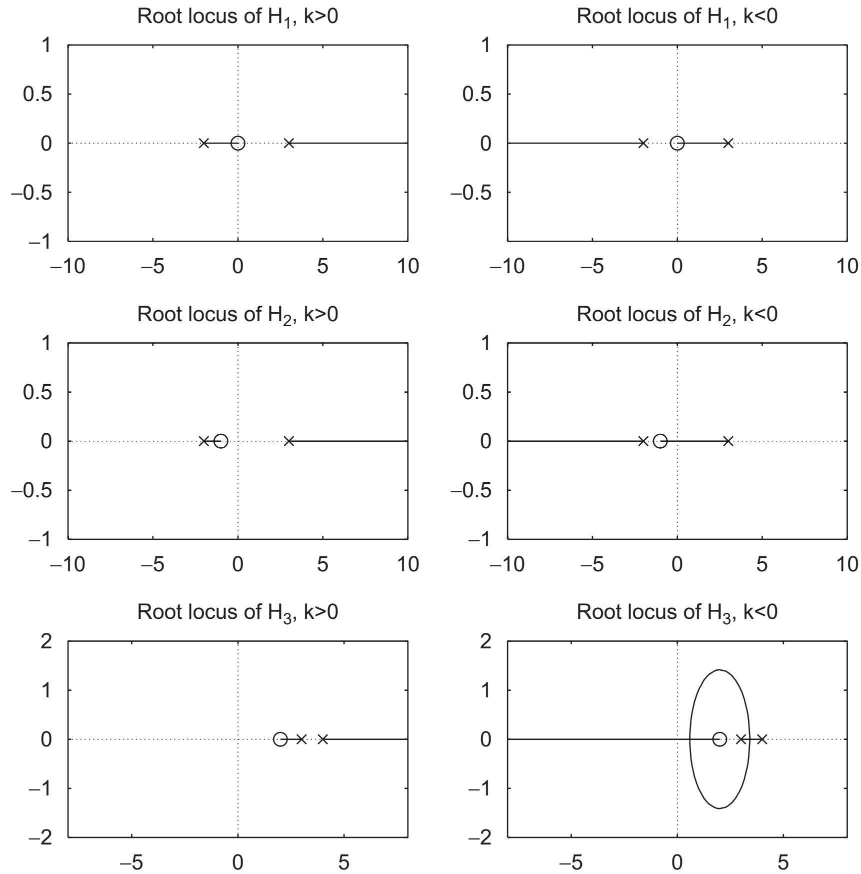

Fig. 2. Root loci for $H_1(s)$, $H_2(s)$, and $H_3(s)$. The root loci for positive k are depicted on the left, while the root loci for negative k are depicted on the right.

In both cases, the transfer function $C(sI - A_i)^{-1}B = 1/(s + 1)$, from which it is clear that the root locus lies along the negative real line for an appropriately chosen value of the gain k. However, since the root locus of the entire system is given by the root locus of the transfer function united with the fixed, unreachable modes, only the second system is stabilizable in this case since the unreachable mode lies in the open left half plane. The first system has a root locus which is real for all k, but an unstable eigenvalue at $s = 1$ always exists. Hence, no feedback of the form $u(x_1, x_2) = v(x_1, x_2)Cx$ can stabilize the first system for any $v(x_1, x_2)$.

6. Conclusion and future work

Some remarks are in order. First, the switching law presented in this paper is not implementable from a practical standpoint since the value of the gain is constant everywhere except on a measure zero set. Nevertheless, it can be shown that by extending the gain used on the stable manifold to an entire cone within the state-space, one can derive control laws which are robust with respect to time delays (see [12] for details).

Second, even though it may appear that the results here are highly dependent on the simplistic nature of second order systems, extensions of this work to higher dimensional systems do exist. Indeed, [12] presents an example where the control laws here are used to develop a stabilizing controller for a fourth-order system by relying on standard results of perturbation theory. Furthermore, the results here can be weakly extended to a sufficient condition for stabilizability in higher dimensions if the root locus of the LTI plant admits only *one* unstable mode for a particular selection of the feedback gain.

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