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Input/Output Stability of Systems Over Finite Alphabets

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Abstract— **This paper considers systems over finite alphabets, that is discrete-time systems whose input and output signals take their values in finite sets. Three notions of input/output stability are proposed: gain stability, incremental stability and external stability. Sufficient conditions for stability of feedback interconnections of stable systems, in the finite gain and incremental sense, are derived. Simple examples are used to illustrate the practical significance of these notions of stability as performance objectives and as robustness measures.**

I. INTRODUCTION

The terminology 'systems over finite alphabets' is used to refer to discrete-time dynamical systems whose input and output signals take their values in finite sets. These discrete values can be interpreted as quantized versions of analog signals, symbols, or a mix of the two. Many practical systems fall into the general class of systems over finite alphabets, either by virtue of physical limitations on their sensors and actuators resulting in quantization and saturation effects, or due to a logic interface such as programmable logic controllers.

We are interested in developing a robust control framework, modeled after the classical robust control framework [10], where the nominal models and the resulting controllers are finite state machines. The paradigm would be to take a system over finite alphabets, represent it as the feedback interconnection of a finite state machine (the nominal model) and an uncertainty block, described by some integral constraints (gain bounds, for example). A finite state machine controller would be designed based on the nominal model so that the nominal closed loop system meets some specified performance objective, described again in terms of integral constraints such as a gain bound. Robust performance of the actual closed loop system would then be verified using a small gain theorem.

While some approaches do exist for stabilizing or controlling some classes of systems over finite alphabets [2] [5] [6], the proposed framework would provide a unified and systematic way of dealing with such systems. On the other hand, there are classes of systems, typically highly nonlinear and possibly involving hybrid dynamics, for which we do not have a good approach for analysis or controller synthesis. The development of such a framework could potentially lead to some breakthrough, whereby we would impose artificial constraints on the inputs and outputs of such a system in order to reduce it to a problem that is amenable to a systematic treatment using this approach.

In order for the proposed framework to be practically useful, the following three requirements would have to be met: (i) the class of finite state machine models proves to be a viable class of nominal models for systems over finite alphabets whose internal dynamics are highly nonlinear and/or mixed continuous and discrete, (ii) analysis of and controller synthesis for finite state machines can be made computationally tractable, and (iii) physically meaningful performance objectives can be formulated in terms of integral constraints on the signals of the nominal model, and verification of robust performance can be systematically carried out.

Preliminary results addressing (i) and (ii) above were presented in [7] and [8]. In this paper, the focus is solely on (iii). In particular, we consider stability analysis of systems over finite alphabets. An input/output view of this class of systems, which we formally define, is adopted. Three notions of stability are presented and motivated by practical examples. Two small gain theorems are then derived, describing sufficient conditions to guarantee stability of a feedback interconnection of two stable systems.

The following notation is used throughout the paper: given two sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\times\mathcal{B}$ denotes their cartesian product. $\mathbf{Z}_+$ and $\mathbf{R}_+$ denote the set of non-negative integers and the set of non-negative reals, respectively. $\mathcal{A}^{\mathbf{Z}_+}$ is the set of all infinite sequences over set $\mathcal{A}$: that is, $\mathcal{A}^{\mathbf{Z}_+} = \{h : \mathbf{Z}_+ \rightarrow \mathcal{A}\}$. An element of $\mathcal{A}$ is denoted by a while an element of $\mathcal{A}^{\mathbf{Z}_+}$ is denoted by $\mathbf{a}$ or $\{a(t)\}_{t=0}^{\infty}$.

II. SYSTEMS AND STABILITY

A. Systems over Finite Alphabets

We begin by defining a discrete-time system over finite alphabets. A discrete-time signal is understood to be an infinite sequence over some prescribed set, which we refer to as an *alphabet set*. The case where signals are defined over finite alphabet sets is of particular interest in this paper.

Definition 1: A *discrete-time system* S is a set of pairs of signals, $\mathcal{D} \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$.

A system is thus a process characterized by its *feasible signals set* $\mathcal{D}$, which is simply a list of ordered pairs of all the signals (sequences over input alphabet set $\mathcal{U}$) that can be applied as an input to this process, and all the output signals (sequences over output alphabet set $\mathcal{Y}$) that can be potentially exhibited by the process in response to each of the input signals. In particular, when the alphabet sets

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$\mathcal{U}$ and $\mathcal{Y}$ are finite, the systems in question are *systems over finite alphabets*. The notation S and $\mathcal{D}$ will be used interchangeably throughout the paper to denote the system over finite alphabets S characterized by its feasible signals set $\mathcal{D}$.

Remark 1: Throughout the paper, we will refer to a discrete-time system over finite alphabets as a 'system' for short.

B. Notions of Input/Output Stability

Three notions of input/output stability for systems over finite alphabets are proposed in this section: gain stability, incremental stability and external stability.

Notions of finite gain stability are extensively used in the classical robust control framework. In that setting, the signal spaces are vector spaces, and the gain of an LTI system is interpreted as an induced norm, representing the maximum amplification of an input signal acted on by the system, in some signal norm of choice and assuming that the system starts from zero initial conditions. A more general definition is needed when the models are not LTI and/or when the signal sets do not have a vector space structure. In what follows, we propose a definition of finite gain stability relevant to the systems of interest to us.

Definition 2: Let $\rho : \mathcal{U} \to \mathbf{R}$ and $\mu : \mathcal{Y} \to \mathbf{R}$ be given functions. A system S is *ρ/μ gain stable* if there exists a finite non-negative constant γ such that the following inequality is satisfied for every pair $(\mathbf{u}, \mathbf{y})$ in $\mathcal{D}$:

$$\inf_{T\geq 0} \sum_{t=0}^{T} \gamma\rho(u(t)) - \mu(y(t)) > -\infty \tag{1}$$

Of particular interest is the case where ρ and μ are zero on some $\mathcal{U}_o \subset \mathcal{U}$ and $\mathcal{Y}_o \subset \mathcal{Y}$, respectively, and strictly positive elsewhere.

Definition 3: Let $\rho : \mathcal{U} \to \mathbf{R}_+$ and $\mu : \mathcal{Y} \to \mathbf{R}_+$ be given non-negative functions. The *ρ/μ gain of S* is the greatest lower bound of γ such that (1) is satisfied.

It has long been recognized that finite gain stability, while useful in an LTI setting, is often too weak to be a useful property for general nonlinear systems within a robust analysis framework (the two notions are equivalent for LTI systems). Various incremental notions of input/output stability have been proposed and studied as possible alternatives applicable to nonlinear systems within a small gain theorem setting [3], [1], [4], [9]. The second notion of input/output stability considered in this paper is a notion of incremental stability, defined below. A real valued function $d : \mathcal{A}\times\mathcal{A} \to \mathbf{R}$ is said to be *symmetric* if $d(a, b) = d(b, a)$, $\forall a, b \in \mathcal{A}$ and *positive definite* if:

$$d(a, b) \geq 0,\ \forall a, b \in \mathcal{A} \text{ and } d(a, b) = 0 \Leftrightarrow a = b$$

Definition 4: A system S is *incrementally stable* if there exists a finite non-negative constant γ and a pair of symmetric positive definite functions, $d_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \to \mathbf{R}_+$ and $d_{\mathcal{Y}} : \mathcal{Y}\times\mathcal{Y} \to \mathbf{R}_+$, such that for any two pairs $(\mathbf{u}_1, \mathbf{y}_1)$ and $(\mathbf{u}_2, \mathbf{y}_2)$ in $\mathcal{D}$, the following inequality is satisfied:

$$\inf_{T\geq 0} \sum_{t=0}^{T} \gamma d_{\mathcal{U}}(u_1(t), u_2(t)) - d_{\mathcal{Y}}(y_1(t), y_2(t)) > -\infty \tag{2}$$

Given a particular choice of symmetric positive definite functions $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$, the greatest lower bound of γ such that (2) is satisfied is called the *$d_{\mathcal{U}}/d_{\mathcal{Y}}$ incremental gain of S*.

The final notion of stability, external stability, was introduced in [7] for a special class of systems, within the context of an approximation problem. Informally, a system is externally stable if it appears to forget its past. The following definition of external *instability* makes this notion rigorous.

Definition 5: A system S is *externally unstable* if there exists a finite constant $\tau \geq 0$ and two elements $(\mathbf{u}, \mathbf{y}_1)$ and $(\mathbf{u}, \mathbf{y}_2)$ of $\mathcal{D}$ such that $y_1(t') \neq y_2(t')$ for some $t' \in [t, t+\tau]$, for every $t \geq 0$.

C. Some Comments on Input/Output Stability

Remark 2: In instances where the alphabet sets have some particular algebraic structure, there is a natural choice for $\mathcal{U}_o$ and $\mathcal{Y}_o$. For example, for an alphabet set with a monoid structure or a field structure, the natural choice is the singleton consisting of the identity element of the monoid and the additive identity element of the field, respectively.

Remark 3: Let $\rho_1 : \mathcal{U} \to \mathbf{R}_+$ and $\rho_2 : \mathcal{U} \to \mathbf{R}_+$ be zero on $\mathcal{U}_o \subset \mathcal{U}$ and strictly positive elsewhere. Let $\mu_1 : \mathcal{Y} \to \mathbf{R}_+$ and $\mu_2 : \mathcal{Y} \to \mathbf{R}_+$ be zero on $\mathcal{Y}_o \subset \mathcal{Y}$ and strictly positive elsewhere. Set

$$c_\rho = \max_{u\in\mathcal{U}-\mathcal{U}_o} \frac{\rho_1(u)}{\rho_2(u)}$$

and

$$c_\mu = \min_{y\in\mathcal{Y}-\mathcal{Y}_o} \frac{\mu_1(y)}{\mu_2(y)}$$

The following inequality holds for any non-negative constant γ, and any $T \geq 0$:

$$c_\mu \sum_{t=0}^{T} \frac{\gamma c_\rho}{c_\mu} \rho_2(u(t)) - \mu_2(y(t)) \geq \sum_{t=0}^{T} \gamma\rho_1(u(t)) - \mu_1(y(t))$$

It follows from this inequality and from finiteness of the alphabet sets that if, for some choice of functions ρ_1 and μ_1 zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$ respectively and positive elsewhere, there exists a non-negative constant γ, say $\gamma = \gamma_1$, such that (1) holds; then for any other choice of functions ρ_2 and μ_2 zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$ respectively and positive elsewhere, there exists a value $\gamma_2 \geq 0$, in particular $\gamma_2 = \frac{c_\rho\gamma_1}{c_\mu}$, such that (1) also holds for all $(\mathbf{u}, \mathbf{y}) \in \mathcal{D}$. Thus, given a system S and a particular choice of $\mathcal{U}_o$ and $\mathcal{Y}_o$, the existence (or non-existence) of a finite γ and functions $\rho : \mathcal{U} \to \mathbf{R}_+$ and $\mu : \mathcal{Y} \to \mathbf{R}_+$, zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$ respectively and positive elsewhere, such that (1) is satisfied is an intrinsic property of the system. When such a γ exists, the system is said to be *gain stable about* $(\mathcal{U}_o, \mathcal{Y}_o)$.

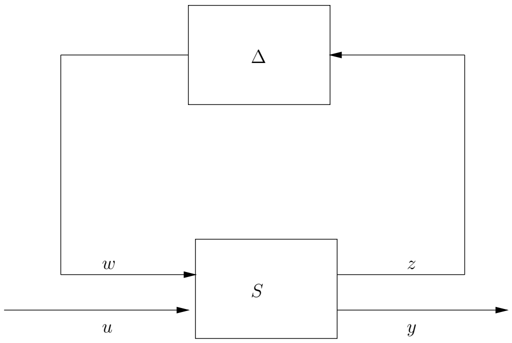

Fig. 1. Feedback interconnection of S and Δ

Remark 4: By an argument similar to that made in Remark 3, it is clear that incremental stability (or lack of it) is an intrinsic property of a given system. However, the numerical value of the incremental gain of a stable system depends on the choice of functions $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$.

Remark 5: Incremental stability is a stronger notion than external stability. To see that, suppose that a system S is incrementally stable. Then for any pair of elements $(\mathbf{u}, \mathbf{y}_1)$ and $(\mathbf{u}, \mathbf{y}_2)$ in $\mathcal{D}$, we have the following inequality:

$$\sup_{T \geq 0} \sum_{t=0}^{T} d_{\mathcal{Y}}(y_1(t), y_2(t)) < \infty$$

Since function $d_{\mathcal{Y}}$ only takes on a finite number of values ($\mathcal{Y}$ is finite), the above inequality allows us to conclude that the system is externally stable.

III. STABILITY OF FEEDBACK INTERCONNECTIONS OF STABLE SYSTEMS

The feedback interconnection of two systems S and Δ as shown in Figure 1 is a system with feasible signals set $\mathcal{D} \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$:

$$\mathcal{D} = \{(\mathbf{u}, \mathbf{y}) | \exists \mathbf{w}, \mathbf{z} \text{ s.t. } ((\mathbf{u}, \mathbf{w}), (\mathbf{y}, \mathbf{z})) \in \mathcal{D}_S, (\mathbf{z}, \mathbf{w}) \in \mathcal{D}_\Delta\}$$

Consider the feedback interconnection of S and Δ and suppose that both systems are stable in the same sense (either gain stable or incrementally stable). The question is, under what conditions can we ensure that the feedback interconnection with input u and output y is stable in the same sense?

Theorem 1: (A Small Gain Theorem) Suppose that system S is ρ_S/μ_S gain stable and satisfies (1) with $\gamma = 1$, for some $\rho_S : \mathcal{U} \times \mathcal{W} \to \mathbf{R}$ and $\mu_S : \mathcal{Y} \times \mathcal{Z} \to \mathbf{R}$. Suppose also that system Δ is ρ_Δ/μ_Δ gain stable and satisfies (1) with $\gamma = 1$, for some $\rho_\Delta : \mathcal{Z} \to \mathbf{R}$ and $\mu_\Delta : \mathcal{W} \to \mathbf{R}$. The interconnected system (S, Δ) with input u and output y is ρ/μ gain stable for $\rho : \mathcal{U} \to \mathbf{R}$ and $\mu : \mathcal{Y} \to \mathbf{R}$ defined by:

$$\rho(u) \doteq \max_{w \in \mathcal{W}} \{\rho_S(u, w) - \mu_\Delta(w)\} \tag{3}$$

$$\mu(y) \doteq \min_{z \in \mathcal{Z}} \{\mu_S(y, z) - \rho_\Delta(z)\} \tag{4}$$

and satisfies (1) with $\gamma = 1$. □

Proof: By assumption, all feasible signals of S satisfy:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_S(u(t), w(t)) - \mu_S(y(t), z(t)) > -\infty \tag{5}$$

and those of Δ satisfy:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho_\Delta(z(t)) - \mu_\Delta(w(t)) > -\infty \tag{6}$$

For functions ρ and μ defined in (3) and (4), (5) implies that all feasible signals of system S satisfy the following condition:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho(u(t)) + \mu_\Delta(w(t)) - \mu(y(t)) - \rho_\Delta(z(t)) > -\infty \tag{7}$$

Adding (6) to (7), and noting that the infimum of the sum of two functions is larger than or equal to the sum of the infimums of the functions, we get:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \rho(u(t)) - \mu(y(t)) > -\infty \tag{8}$$

Hence, the interconnected system (S, Δ) is ρ/μ gain stable and satisfies (1) with $\gamma = 1$. ■

In particular, we may be interested in proving stability of the interconnection (S, Δ) about $(\mathcal{U}_o, \mathcal{Y}_o)$, for some specific choice of $\mathcal{U}_o \subset \mathcal{U}$ and $\mathcal{Y}_o \subset \mathcal{Y}$. Theorem 1 allows us to verify this if ρ and μ defined in (3) and (4) satisfy the requirement that they're zero on $\mathcal{U}_o$ and $\mathcal{Y}_o$, respectively, and strictly positive otherwise.

Remark 6: An interesting special case is when all the alphabet sets are finite subsets of $\mathbf{R}$, and gain stability of systems S and Δ are interpreted as l_2 gain conditions in $\mathbf{R}$. In this case, we have: $\rho_S(u, w) = |u|^2 + |w|^2$, $\mu_S(y, z) = |y|^2 + |z|^2$, $\mu_\Delta(w) = |w|^2$, $\rho_\Delta(z) = |u|^2$, and consequently $\rho(u) = |u|^2$ and $\mu(y) = |y|^2$. Our formulation thus reduces to the standard small gain result: if each of S and Δ are stable with l_2 gain not exceeding 1, then so is their interconnection.

Theorem 2: (An Incremental Small Gain Theorem) Suppose that system S is incrementally stable with $d_{\mathcal{U}_S}/d_{\mathcal{Y}_S}$ incremental gain not exceeding 1, for some symmetric positive definite functions $d_{\mathcal{U}_S} : (\mathcal{U} \times \mathcal{W}) \times (\mathcal{U} \times \mathcal{W}) \to \mathbf{R}_+$ and $d_{\mathcal{Y}_S} : (\mathcal{Y} \times \mathcal{Z}) \times (\mathcal{Y} \times \mathcal{Z}) \to \mathbf{R}_+$. Suppose also that system Δ is incrementally stable with $d_{\mathcal{U}_\Delta}/d_{\mathcal{Y}_\Delta}$ incremental gain not exceeding 1, for some symmetric positive definite functions $d_{\mathcal{U}_\Delta} : \mathcal{Z} \times \mathcal{Z} \to \mathbf{R}_+$ and $d_{\mathcal{Y}_\Delta} : \mathcal{W} \times \mathcal{W} \to \mathbf{R}_+$. If symmetric functions $d_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \to \mathbf{R}$ and $d_{\mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \to \mathbf{R}$ given by:

$$d_{\mathcal{U}}(u_1, u_2) \doteq \max_{w_1, w_2} \{d_{\mathcal{U}_S}((u_1, w_1), (u_2, w_2)) - d_{\mathcal{Y}_\Delta}(w_1, w_2)\} \tag{9}$$

$$d_{\mathcal{Y}}(y_1, y_2) \doteq \min_{z_1, z_2} \{d_{\mathcal{Y}_S}((y_1, z_1), (y_2, z_2)) - d_{\mathcal{U}_\Delta}(z_1, z_2)\} \tag{10}$$

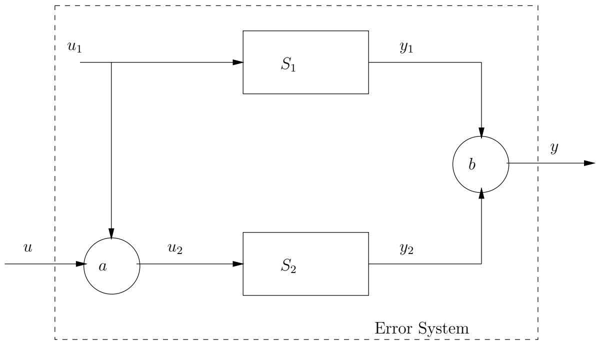

Fig. 2. The error system of S_1 and its approximation, S_2

are positive definite, the interconnected system (S, Δ) with input u and output y is incrementally stable, and its $d_{\mathcal{U}}/d_{\mathcal{Y}}$ incremental gain does not exceed 1. □

Proof: By assumption, all feasible signals of S satisfy:

$$\inf_{T\geq 0} \sum_{t=0}^{T} \{d_{\mathcal{U}_S}((u_1(t), w_1(t)), (u_2(t), w_2(t))) - d_{\mathcal{Y}_S}((y_1(t), z_1(t)), (y_2(t), z_2(t)))\} > -\infty \tag{11}$$

and those of Δ satisfy:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_{\mathcal{U}_\Delta}(z_1(t), z_2(t)) - d_{\mathcal{Y}_\Delta}(w_1(t), w_2(t)) > -\infty \tag{12}$$

For functions $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$ defined by (9) and (10), (11) implies that:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_{\mathcal{U}}(u_1(t), u_2(t)) + d_{\mathcal{Y}_\Delta}(w_1(t), w_2(t)) - d_{\mathcal{Y}}(y_1(t), y_2(t)) - d_{\mathcal{U}_\Delta}(z_1(t), z_2(t)) > -\infty \tag{13}$$

Adding (12) and (13), and noting that the infimum of the sum of two functions is larger than or equal to the sum of the infimums of the functions, we get that:

$$\inf_{T\geq 0} \sum_{t=0}^{T} d_{\mathcal{U}}(u_1(t), u_2(t)) - d_{\mathcal{Y}}(y_1(t), y_2(t)) > -\infty \tag{14}$$

If $d_{\mathcal{U}}$ and $d_{\mathcal{Y}}$, which are symmetric by definition, are also positive definite, the interconnection (S, Δ) is incrementally stable and its $d_{\mathcal{U}}/d_{\mathcal{Y}}$ incremental gain does not exceed 1. ■

IV. ILLUSTRATIVE EXAMPLES

Example 1: The system over finite alphabets $\mathcal{U} = \{-1, 0, 1\}$ and $\mathcal{Y} = \{-K, 0, K\}$ defined by its feasible signals set $\mathcal{D} \subset \mathcal{U}^{\mathbf{Z}_+} \times \mathcal{Y}^{\mathbf{Z}_+}$:

$$\mathcal{D} = \{(\mathbf{u}, \mathbf{y}) | y(t) = Ku(t), \forall t \in \mathbf{Z}_+\}$$

is a gain K whose input is restricted to three values: -1, 0 and 1.

The following three simple examples practically illustrate each of the definitions proposed in section II-B.

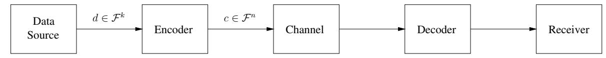

Fig. 3. Coding Setup

Example 2: Let S_1 and S_2 be given systems over binary alphabets $\mathcal{U}_1 = \mathcal{U}_2 = \{\alpha_1, \alpha_2\}$ and $\mathcal{Y}_1 = \mathcal{Y}_2 = \{\beta_1, \beta_2\}$. Suppose that S_2 is a lower complexity system (in some appropriate measure). Typically, S_2 is considered to be a good approximation of S_1 if its response to every input is close to that of S_1 to a similar input. This can be quantified by the ρ/μ gain of the error system (Figure 2) with input $u \in \{0, 1\}$ and output $y \in \{0, 1\}$, where $\rho : \{0, 1\} \to \mathbf{R}_+$ and $\mu : \{0, 1\} \to \mathbf{R}_+$ are simply the identity maps. Function $a : \{\alpha_1, \alpha_2\} \times \{0, 1\} \to \{\alpha_1, \alpha_2\}$ is a “flip in input” transformation defined by:

$$a(u_1, u) \doteq \begin{cases} u_1 & \text{if } u = 0 \\ 1 - u_1 & \text{if } u = 1 \end{cases}$$

Function $b : \{\beta_1, \beta_2\} \times \{\beta_1, \beta_2\} \to \{0, 1\}$ is a “binary comparator” defined by:

$$b(y_1, y_2) \doteq \begin{cases} 0 & \text{if } y_1 = y_2 \\ 1 & \text{otherwise} \end{cases}$$

A finite ρ/μ gain indicates that the response of S_1 and S_2 to the same input can only differ by a finite number of terms. The numerical value of the gain is an indication of the sensitivity of the quality of approximation to input perturbations.

Example 3: Convolutional codes are widely used to add redundancy to data transmitted over noisy channels so as to enable error free decoding at the receiver end (Figure 3). A convolutional encoder is a map $E : (\mathcal{F}^k)^{\mathbf{Z}_+} \to (\mathcal{F}^n)^{\mathbf{Z}_+}$, where $\mathcal{F}$ is a finite field and n and k are integers with $n > k$, such that $\mathcal{C} = E((\mathcal{F}^k)^{\mathbf{Z}_+})$ is a right shift-invariant linear subspace of $(\mathcal{F}^n)^{\mathbf{Z}_+}$. Given a convolutional code $\mathcal{C}$, the problem of finding an encoder for it can be formulated as the problem of finding a state-space realization for an invertible map $\phi_E : \mathcal{C} \to (\mathcal{F}^k)^{\mathbf{Z}_+}$. A good encoder is one that is ‘non-catastrophic’, among other properties. An encoder is said to be catastrophic if two codewords $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}$ differing by a finite number of terms correspond to two data sequences $\mathbf{d}_1, \mathbf{d}_2 \in (\mathcal{F}^k)^{\mathbf{Z}_+}$ differing by an infinite number of terms. Ensuring that the system over finite alphabets S with feasible signals set $\mathcal{D} = \{(\mathbf{c}, \mathbf{d}) \in \mathcal{C} \times (\mathcal{F}^k)^{\mathbf{Z}_+} | \ \mathbf{d} = \phi_E(\mathbf{c})\}$ is incrementally stable, and hence satisfies:

$$\inf_{T\geq 0} \sum_{t=0}^{T} \gamma d_{\mathcal{U}}(c_1(t), c_2(t)) - d_{\mathcal{Y}}(d_1(t), d_2(t)) > -\infty$$

for some finite $\gamma \geq 0$ and some symmetric positive definite functions $d_{\mathcal{U}} : \mathcal{F}^n \times \mathcal{F}^n \to \mathbf{R}_+$ and $d_{\mathcal{Y}} : \mathcal{F}^k \times \mathcal{F}^k \to \mathbf{R}_+$, allows us to ensure that the corresponding encoder ($E = \phi^{-1}$) is non-catastrophic.

Example 4: Consider a stable LTI system described by:

$$x(t+1) = \frac{1}{2}x(t) + u(t)$$

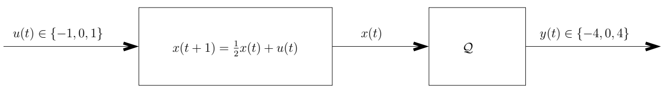

Fig. 4. Stable LTI system with state quantizer

and an output quantizer Q described by:

$$y(x) = \begin{cases} \vdots & \\ -4 & -6 \leq x < -2 \\ 0 & -2 \leq x < 2 \\ 4 & 2 \leq x < 6 \\ \vdots & \end{cases}$$

connected as shown in Figure 4. The input to the system is assumed to be restricted to take on the values 0, ± 1, and the initial state of the LTI system is assumed to lie in the interval $[-6, 6)$. Consequently, the output of Q takes on the values 0, ± 4. Even though the LTI system is stable (pole inside the unit disk), the system with input u and output y is not externally stable: consider the constant input $u(t) = 1$ and the two initial conditions $x_1(0) = 0$ and $x_2(0) = 4$. The corresponding constant outputs, $y_1(t) = 0$ and $y_2(t) = 4$, are unequal at every time step. The lack of external stability in this (or any) system has an important consequence [7]. We cannot expect to find an arbitrarily close approximation for such a system in the traditional sense of Example 2: that is, another system whose response to each input is not very different from that of the original system. For systems that are externally unstable, a different approximation paradigm is needed, in which the initial condition is explicitly estimated.

The typical usage of the small gain theorem (Theorem 1) is as follows: a system with complex dynamics is represented as the feedback interconnection (Figure 5) of a simple model with known dynamics, M (potentially belonging to the class of deterministic finite state machine models, which will be described in the next section) and a system Δ, representing the approximation error and/or modeling uncertainty, described by a gain condition, (6). A physical performance objective is mathematically described as a gain condition (as in (8)), to be satisfied by the controlled plant with input u and output y. A controller K is designed, based on the nominal model M, so that the closed loop system (M, K), with inputs u and w and outputs y and z, is gain stable and satisfies another condition, namely (5), with functions ρ_S and μ_S chosen so that (3) and (4) hold. Theorem 1, with system S representing the feedback interconnection of M and K, then allows us to ensure that the actual closed loop system (S, Δ) satisfies the performance objective, (8). The following simple queuing example illustrates this procedure.

Example 5: Consider a queuing system consisting of a single buffer and m deterministic servers, of which only one can be used at any given time. The i^{th} server operates at a fixed rate r_i and incurs operating cost c_i per unit time, with $r_i < r_j$ and $c_i < c_j$ for $i < j$. Let $a(t)$ be the number of packets arriving at the buffer at time step t, where $t \in \mathbf{Z}_+$.

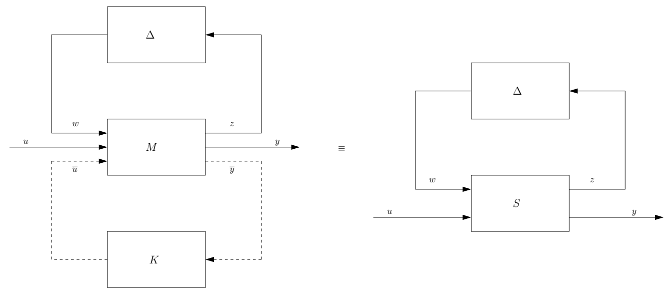

Fig. 5. Setup for Robust Performance Verification

We assume that only one server can be utilized in any one time step, and that there is a physical limitation to how many packets can arrive at any time step (i.e. an upper bound on $a(t)$). A controller is a system that implements a control law which maps the length of the queue in the buffer to a choice of server to be used. The queuing system is said to be stable if the queue size remains finite at all times. A typical analysis question is the following: given a queuing system, a controller and some limited knowledge about the arrival process, verify that the resulting queuing system is stable. Assume, for instance, that the controller picks the same server, with rate r^o, regardless of the length of the queue and that the arrival process obeys the Leaky Bucket model, namely:

$$A(s, t) \leq \alpha \cdot (t - s) + \beta$$

where $A(s, t)$ is the total number of (integer valued) arrivals in time interval $[s, t]$ and α and β are given positive constants. We wish to find conditions under which we can guarantee stability of this queuing system. The system can be modeled as a system over finite alphabets (system 'M' in figure 5) with two inputs, the server rate r (control input '$\overline{u}$' in the figure) and the number of arrivals a (disturbance input 'w' in the figure), where $r(t) \in \mathcal{R} = \{r_1, \ldots, r_m\}$ and $a(t) \in \mathcal{A} = \{0, 1, \ldots, \beta\}$. Exogenous input u can be thought of as being identically 0 here. The internal state of M is q, the length of the queue, described by the following update equation:

$$q(t+1) = \max\{0, q(t) + a(t) - r(t)\}$$

The system has output q (sensor output '$\overline{y}$' in the figure), r (input to the uncertainty block Δ, 'z' in the figure) and δq (output 'y' used to characterize the performance objective in the figure), where $\delta q(t) = q(t+1) - q(t)$. The arrival process can be modeled by an uncertainty block Δ satisfying the gain condition:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \Big(\frac{\alpha}{r^o} r(t) - a(t) \Big) > -\infty$$

The following gain condition is satisfied by M:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \Big(- a(t) + (\delta q(t) + r(t)) \Big) > -\infty$$

We wish to verify that the actual closed loop system satisfies the stability requirement that $q(T) < \infty, \forall T$. Assuming (reasonably) that the initial length of the queue is finite, this can be equivalently expressed as:

$$\inf_{T \geq 0} \sum_{t=0}^{T} -\delta q(t) > -\infty \tag{15}$$

In the terminology of Theorem 1, we have $\rho_S(u,a) = -a$, $\mu_S(\delta q, r) = \delta q + r$, $\rho_\Delta(r) = \frac{\alpha}{r^o} r$ and $\mu_\Delta(a) = a$. For $\rho(u) = \max_{a \in \mathcal{A}} \{-2a\} = 0$ and $\mu(\delta q) = \min_{r \in \{r^o\}} \{\delta q + r - \frac{\alpha}{r^o} r\} = \delta q + (r^o - \alpha)$, Theorem 1 allows us to write that:

$$\inf_{T \geq 0} \sum_{t=0}^{T} \Big(-\delta q(t) - (r^o - \alpha) \Big) > -\infty$$

which implies (15) if $r^o - \alpha \geq 0$. Thus, a sufficient condition for stability is $r^o \geq \alpha$.

The usage of the incremental small gain theorem is similar. The choice between a gain condition or an incremental gain condition to represent the performance objective depends on the specifics of the problem.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

Three input/output notions of stability were proposed for systems over finite alphabets. Simple examples were presented to illustrate these notions of stability and their relevance in describing practical performance objectives. Two small gain theorems giving sufficient conditions for the interconnection to be stable assuming each of the components is stable were derived, thus showing that these notions of stability are useful as measures of robustness within this setting.

B. Future Works

Future work will focus on two directions:

1) The examples presented in this paper were chosen for their simplicity and their ability to illustrate the definitions and theorems presented. As stated in the introduction, there are classes of systems, generally highly nonlinear or hybrid, for which no good analysis or synthesis methods exist. We will consider examples falling in this category in order to demonstrate the practical impact of this approach.
2) The problem of verifying stability or computing the gain of a given system over finite alphabets was not addressed in this paper. In [8], we presented stability verification algorithms for a class of nominal models of interest, deterministic finite state machines, that were polynomial in the number of states of the model. We plan to focus further on this class of models with the goal of finding ways to exploit structural properties, whether arising from specific interconnections or from algebraic structure imposed on the internal dynamics of the model and/or on the signal sets, to further reduce the computational complexity of this problem.

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