Feedback Stabilization of Uncertain Systems in the Presence of a Direct Link

Nuno C. Martins, *Member, IEEE*, Munther A. Dahleh, *Fellow, IEEE*, and Nicola Elia, *Member, IEEE*

***Abstract*—We study the stabilizability of uncertain stochastic systems in the presence of finite capacity feedback. Motivated by the structure of communication networks, we consider a variable rate digital link. Such link is used to transmit state measurements between the plant and the controller. We derive necessary and sufficient conditions for internal and external stabilizability of the feedback loop. In accordance with previous publications, stabilizability of unstable plants is possible if and only if the link's average transmission rate is above a positive critical value. In addition, stability in the presence of uncertainty in the plant is analyzed using a small-gain argument. We also show that robustness can be increased at the expense of a higher transmission rate.**

***Index Terms*—Control over networks, limited information, robust stability.**

I. Introduction

WITH A wide range of formulations, control in the presence of communication constraints has been the focus of intense research. The need to remotely control one or more systems from a central location, has stimulated the study of stabilizability of unstable plants when the information flow in the feedback loop is finite. Such limitation results from the use of an analog communication channel or a network digital link as a way to transmit information about the state of the plant. It can also be viewed as an abstraction of the computational constraints created by several systems sharing a common decision center.

Various publications in this field have introduced necessary and sufficient conditions for the stabilizability of unstable plants in the presence of data-rate constraints. The construction of a stabilizing controller requires that the data-rate of the feedback loop is above a nonzero critical value [13], [21], [23]–[25], [27]. Different notions of stability have been investigated, such as containability [29], [30], moment stability [17], [18], [21], and stability in the almost sure sense [26]. The last two are different when the state is a random variable. That happens when disturbances are random or if the communication link is stochastic. In [25] and [26], it is shown that the necessary and sufficient condition for almost sure stabilizability of finite dimensional linear and time-invariant systems is given by an inequality of the type $\mathcal{C} > \mathcal{R}$. The parameter $\mathcal{C}$ represents the average data-rate at which information can be reliably transmitted through the feedback loop and $\mathcal{R}$ is a quantity that depends on the eigenvalues of A, the dynamic matrix of the system. Different notions of stability may lead to distinct requirements for stabilization. For tighter notions of stability, such as in the mth moment sense, the knowledge of $\mathcal{C}$ may not suffice. More informative notions, such as higher moments or any-time capacity [21], are necessary. Results for the problem of state estimation in the presence of information constraints can be found in [22], [29], and [12].

A. Main Contributions of this Paper

In this paper, we study the moment stabilizability of uncertain stochastically time-varying systems, assuming that the feedback loop uses a digital communication link with a stochastically time-varying rate. In contrast with [16], we consider systems whose time-variation is governed by an identically and independently distributed (i.i.d.) process which may be defined over a continuous and unbounded alphabet. We also provide complementary results to [7], [10], [16], because we adopt a more general problem formulation, since we consider external disturbances and uncertainty on the plant and a digital communication link with a stochastically timevarying rate of transmission.[1]. For simplicity of exposition, such link is referred to as a direct link.[2]

We show that robust moment stabilizability requires that the average transmission rate of the direct link must satisfy $C > R + \alpha + \beta$, where $\alpha, \beta \geq 0$ are constants that quantify the influence of randomness at the direct link and at the plant, respectively. As a consequence, C must be higher than R to compensate for randomness both at the plant and at the direct link itself. The conclusion that $C > R$ is not sufficient for moment stabilization, in the presence of a stochastically timevarying rate, was originally derived by [22]. The work of [22] was an important motivation for our work and the treatment of the nominal moment stabilization, using a parameterized notion of capacity, denoted as anytime capacity, can be found there. If the plant and the direct link are deterministic then we get $\beta = 0$ and $\alpha = 0$, which is consistent with the condition $C > R$ derived by [24].

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N. C. Martins is with the Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742-3285 USA (e-mail: nmartins@isr.umd.edu).

M. A. Dahleh is with the Department of Electrical Engineering and Computer Science, the Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

N. Elia is with the Department of Electrical Engineering, Iowa State University, Ames, IA 50010 USA.

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[1]The model for the communication link adopted in this paper is described in full detail in Section II-A

[2]The designation "direct link" is adopted in standard texts such as [19]. A direct link is a digital communication link established between two nodes and it represents the simplest digital communication network. The aforementioned direct link may have a stochastically time-varying rate as a result of fading and collisions[19]

We also show that model uncertainty in the plant can be tolerated. By using an appropriate measure, we prove that an increase in C leads to higher tolerance to uncertainty. All of our conditions for stability are expressed as simple inequalities where the terms depend on the description of uncertainty in the plant as well as the statistics of the system and the direct link. The state feedback stabilization, under a deterministic bit-rate constraint, of linear and time-invariant systems in the absence of external excitation and subject to linear and memoryless uncertainty, can be found in [20]. A different approach to dealing with robustness, with respect to transmission rates, can be found in [11].

Besides the Introduction, this paper has four sections: Section II comprises the problem formulation and preliminary definitions; in Section III we prove sufficiency conditions by constructing stabilizing feedback schemes; a proof of the necessary condition for stability can be found in Section IV; in Section V we give a detailed interpretation of certain quantities introduced in the paper.

The following notation is adopted.

- We reserve $k \in \mathbb{N}$ to represent discrete time and $t \in \mathbb{R}$ to denote continuous time.
- Whenever that is clear from the context we refer to a sequence of real numbers $x(k)$ simply as x. In such cases we may add that $x \in \mathbb{R}^\infty$.
- Random variables are represented using boldface letters, such as $\mathbf{w}$.
- If $\mathbf{w}(k)$ is a stochastic process, then we use $w(k)$ to indicate a specific realization. According to the convention used for sequences, we may denote $\mathbf{w}(k)$ just as $\mathbf{w}$ and $w(k)$ as w.
- The expectation operator over $\mathbf{w}$ is written as $\mathcal{E}[\mathbf{w}]$.
- If E is a probabilistic event, then its probability is indicated as $\mathcal{P}(E)$.
- We write $\log_2(.)$ simply as $\log(.)$.
- The set of causal operators between infinite sequences is represented by $\mathbb{G}$. If $G : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is an element of $\mathbb{G}$, $x \in \mathbb{R}^\infty$ and $y = G(x)$ then $y(k)$ may be written as $G(x)(k)$.
- If $x \in \mathbb{R}^\infty$, then $\|x\|_1 = \sum_{i=0}^{\infty} |x(i)|$ and $\|x\|_\infty = \sup_{i\in\mathbb{N}} |x(i)|$.

Definition 1.1: (Causal Operators With Pre-Defined Memory Length): Let ϱ be a given constant in the set $\mathbb{N}_+ \bigcup \{\infty\}$. The set of causal operators with memory of, at most, ϱ steps is a subset of $\mathbb{G}$ defined as

$$\mathbb{G}_\varrho = \left\{G : \forall x \in \mathbb{R}^\infty, \forall k \in \mathbb{N}_+, G(x)(k) = G\left(\Pi_{k-\varrho}^{k} x\right)(k)\right\} \tag{1}$$

where the projection $\prod_{k-\varrho}^{k}$ is given by

$$\left(\Pi_{k-\varrho}^{k} x\right)(q) \overset{\text{def}}{=} \begin{cases} x(q), & \text{if } q \in [k-\varrho, k], \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Definition 1.2: (Infinity Norm): If $G_f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is an element of $\mathbb{G}$ then we denote its infinity induced norm as $\|G_f\|_\infty$. Formally, $\|G_f\|_\infty$ is computed as

$$\|G_f\|_\infty = sup_{x\neq 0} \frac{\|G_f(x)\|_\infty}{\|x\|_\infty}. \tag{3}$$

II. Problem Formulation

We study the moment stabilizability of uncertain stochastic systems under communication constraints. Motivated by the type of constraints that arise in most computer networks [19], we consider the following class of communication links.

Definition 2.1: (Direct Link): Consider a link that, at every time-step k, transmits $\mathbf{r}(k)$ bits. Such link is well defined provided that $\mathbf{r}(k) \in \{0, \ldots, \bar{r}\}$ is an independent and identically distributed (i.i.d.) random process satisfying

$$\mathbf{r}(k) = \mathcal{C} - \mathbf{r}^\delta(k) \tag{4}$$

where $\mathcal{E}[\mathbf{r}^\delta(k)] = 0$ and $C \geq 0$. The term $\mathbf{r}^\delta(k)$ represents a fluctuation in the transfer rate of the link. More specifically, the link is a stochastic truncation operator $\mathcal{F}_k^l : \{0,1\}^{\bar{r}} \to \bigcup_{i=0}^{\bar{r}} \{0,1\}^i$ defined as

$$\mathcal{F}_k^l(b_1, \ldots, b_{\bar{r}}) = \left(b_1, \ldots, b_{\mathbf{r}(k)}\right), \qquad b_i \in \{0,1\} \tag{5}$$

A. Further Remarks on the Model of the Direct Link and the Stochastic Truncation Operator

We start by providing a short overview of the main features of the most basic network link, which according to [19] is denoted as *direct link*. Subsequently, we explain how the stochastic truncation operator of Definition 2.1 is a natural abstraction for a direct link. In [19, Ch. 2] one can find a complete exposition of this subject, including the historical perspective and nomenclature. Our discussion is entirely based on [19], which follows the *open systems interconnection (OSI)* architecture.

At the physical level, a *direct link* connects two nodes without any routing or intermediate nodes and, for that reason, such connection is considered the building block of any network. The hardware used depends on the transmission medium, but, at the software level, communication devices are abstracted as links that are capable of transmitting messages comprising a finite sequence of bits (frame). The protocols that regulate the communication through direct links are implemented in several layers according to the OSI architecture. Among the tasks of a communication protocol, the following are the most relevant.

- **Framing**: Process of breaking a very large message into a sequence of frames.
- **Error detection**: Using various techniques, errors within each frame are detected and, if possible, are corrected. If correction is not possible then the entire frame is discarded. The probability that a validated frame has errors is negligible and, in practice, one can assume that a frame either goes through without errors or is discarded (erasure).
- **Reliable transmission**: By means of appropriate algorithms, the sender and the receiver nodes use a scheme of acknowledge signals and counters that guarantee message delivery in the presence of frame erasure.
- **Access control**: Avoid conflicts when more than one node wants to send and/or receive information through the same link.

Although we are considering the stabilization of discrete time plants, communication must be planned in continuous time. If

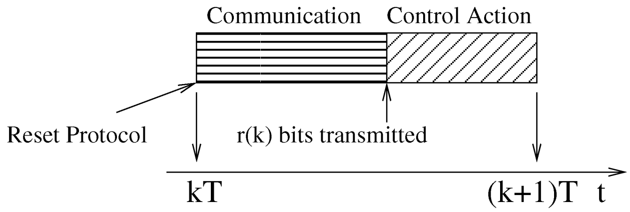

Fig. 1. Control action and communication inter-sample timeline.

we assume that the discrete time plant represents the dynamics of a continuous-time plant at sampling instants $t = kT$, then the time-line depicted in Fig. 1 shows how the intersample time $[kT, (k+1)T[$ can be partitioned to accommodate communication and control action.

At the direct link, we assume the use of the *sliding window* algorithm [19], which is one of the most widely known algorithms for reliable transmission. If we reset the sliding window algorithm at every sampling time $t = kT$ and keep it running during the subsequent communication sub-interval (see Fig. 1) then the following features are important in the motivation of our model.

- **Frame order and flow control**: At every sampling time $t = kT$, the algorithm is reset and the sender submits a new sequence of frames for transmission. Within each communication sub-interval (see Fig. 1), the successfully transmitted frames are available at the receiver sequentially,[3] i.e., if frame number N is delivered then so were all the frames from 1 to $N-1$.
- **Synchronization**: During each communication subinterval, starting at the sampling times $t = kT$, the sender is kept informed of which is the highest N such that frames numbered from 1 to N have arrived at the receiver. This is done by means of ACK signals sent by the receiver according to the process described in [19, Sec. 2.5.2], where such highest N is denoted as *SeqNumT oAck*. We assume that the frame transmission process stops at the end of the communication subinterval and that the last ACK signal is always detected at the sender because the receiver can use the whole control subinterval to send the last ACK signal multiple times.

Based on the previous overview, the following assumptions, together with the properties of the sliding window algorithm, guarantee that the truncation operator is a suitable abstraction.

- Once a sequence of bits $(b_1, \ldots, b_{\bar{r}})$ is submitted for transmission, framing must be sequential, i.e., lower indexed bits (more significant) are included in lower indexed frames (transmitted first). According to the *frame order and flow control* properties of the sliding window algorithm, such sequential framing guarantees that more significant bits are transmitted first. Consequently, if not all frames can be transmitted during the communication subinterval then the least significant bits will not be delivered. Such property is captured in the truncation operator.
- Take $\mathbf{r}(k)$ as the number of bits successfully transmitted during the communication sub-interval initiated at $t = kT$. By assuming that the last ACK is always detected then $\mathbf{r}(k)$ is available at the sender before the next sampling instant[4] $t = (k+1)T$.
- Since $\mathbf{r}(k)$ is a function of the number of frames transmitted, its random behavior arises from random time-out, the existence of collisions generated by other nodes or from fading. We assume that such variation in rate, denoted by $\mathbf{r}^{\delta}(k)$ in (4), is an i.i.d. process.

B. Description of Uncertainty in the Plant

Let $\varrho \in \mathbb{N} + \bigcup\{\infty\}$, $\bar{z}_f \in [0,1)$ and $\bar{z}_a \in [0,1)$ be given constants, along with a stochastic process $\mathbf{z}_a$ and an operator G_f satisfying

$$|\mathbf{z}_a(k)| \leq \bar{z}_a \tag{6}$$

$$G_f \in \mathbb{G}_{\varrho} \text{ and } \|G_f\|_{\infty} \leq \bar{z}_f. \tag{7}$$

Given $x(0) \in [-(1/2), 1/2]$ and $\bar{d} \geq= 0$, we study the existence of stabilizing feedback schemes for the following perturbed plant:

$$\mathbf{x}(k+1) = \mathbf{a}(k)\left(1 + \mathbf{z}_a(k)\right)\mathbf{x}(k) + \mathbf{u}(k) + G_f(\mathbf{x})(k) + \mathbf{d}(k) \tag{8}$$

where $|\mathbf{d}(k)| \leq \bar{d}$, while $\mathbf{z}_a$ and $G_f(\mathbf{x})$ satisfy (6) and (7). Notice that $\mathbf{z}_a(k)$ may represent uncertainty in the knowledge of $\mathbf{a}(k)$, while $G_f(\mathbf{x})(k)$ may portray the output of a feedback uncertainty block G_f. We chose this structure because it allows the representation of a wide class of model uncertainty. It is also allows the construction of a suitable stabilizing scheme.

Example 2.1: If $G_f(\mathbf{x})(k) = \sum_{i=0}^{n-1} \mu_i \mathbf{x}(k-i)$ then $\|G_f\|_{\infty} = \sum_{i=0}^{n-1} |\mu_i|$.

In general, the operator G_f may be nonlinear and timevarying.

Notice that the constant ϱ is an upper-bound on the memory of the allowable uncertainty G_f. The reason why we require such prior knowledge of ϱ is that, in the presence of a stochastic $\mathbf{r}(k)$ or stochastic $\mathbf{a}(k)$, we were not able to guarantee robust stability for infinite memory G_f by just imposing conditions on its induced norms. On the other hand, if the plant and the link are deterministic then we can guarantee robust stability even if ϱ is infinity. These results are stated in full detail and proved in Section III.

C. Statistical Description of $\mathbf{a}(k)$

The process $\mathbf{a}(k)$ is i.i.d. and independent of $\mathbf{r}(k)$, meaning that it carries no information about the link nor the initial state. In addition, for convenience, we use the same representation as in (4) and write

$$\log\left(|\mathbf{a}(k)|\right) = \mathcal{R} + \mathbf{l}_a^{\delta}(k) \tag{9}$$

where $\mathcal{E}[\mathbf{l}_a^{\delta}(k)] = 0$ and $a(k) \neq 0$. Notice that $\mathbf{l}_a^{\delta}(k)$ is responsible for the stochastic behavior, if any, of the plant.

[3]This feature is described in page 109 of [19] where one can read:"... the receiver just makes sure that it does not pass a frame up to the next-higher-level protocol until it has already passed up all the frames with a smaller sequence number."

[4]As we will show in Section III, our results hold if the sender has access to $r(k)$ at $t = (k+1)T$ and not before.

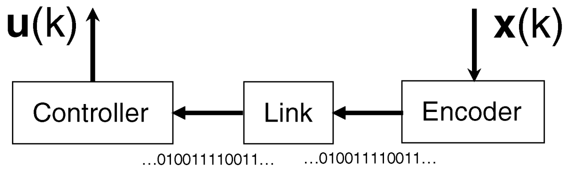

Fig. 2. Structure of the feedback interconnection.

D. Functional Structure of the Feedback Interconnection

We assume that the feedback loop has the structure depicted in Fig. 2, also referred to as the information pattern [28]. The blocks denoted as encoder and controller are stochastic operators whose domain and image are uniquely determined by the diagram. At any given time k, we assume the following.

- The controller has access to the realization $a(0),\ldots,a(k)$. Notice that this assumption does not presuppose that the controller knows the exact realization of the model of the system. According to (8), the dynamics of the system is represented by $\mathbf{a}(k)$ together with a multiplicative and an additive uncertain terms. The main purpose of such uncertain representation is to account for the incomplete knowledge that the controller might have about the plant. The fact that the controller has some knowledge of the model dynamics of a stochastically time-varying plant is a common assumption, for instance, in jump-linear systems [4].
- The encoder and the controller have access to $r(0),\ldots,r(k-1)$. This assumption is motivated in Section II-A, where we explain how typical protocols regulate the reliable transmission of information over a direct link.
- The encoder and the controller have access to the constants ϱ, $\bar{z}_f$, $\bar{z}_a$ and $\bar{d}$.

The encoder and the controller are described as follows.

- The encoder is a function $\mathcal{F}_k^e : \mathbb{R}^{k+1} \to \{0,1\}^{\bar{r}}$ that has the following dependence on observations:

$$\mathcal{F}_k^e\left(\mathbf{x}(0),\mathbf{x}(1),\ldots,\mathbf{x}(k)\right) = (\mathbf{b}_1,\ldots,\mathbf{b}_{\bar{r}}). \tag{10}$$

- The control action results from a map $\mathcal{F}_k^c : \bigcup_{i=0}^{\bar{r}}\{0,1\}^i \to \mathbb{R}$ exhibiting the following functional dependence:

$$\mathbf{u}(k) = \mathcal{F}_k^c\left(\vec{\mathbf{b}}(k)\right) \tag{11}$$

  where $\vec{b}(k)$ are the bits successfully transmitted through the link, i.e.:

$$\vec{\mathbf{b}}(k) = \mathcal{F}_k^l(\mathbf{b}_1,\ldots,\mathbf{b}_{\bar{r}}) = \left(\mathbf{b}_1,\ldots,\mathbf{b}_{\mathbf{r}(k)}\right) \tag{12}$$

  As such, $\mathbf{u}(k)$ can be equivalently expressed as $\mathbf{u}(k) = (\mathcal{F}_k^c \circ \mathcal{F}_k^l \circ \mathcal{F}_k^e)(\mathbf{x}(0),\mathbf{x}(1),\ldots,\mathbf{x}(k))$

Definition 2.2: (Feedback Scheme): We define a feedback scheme as the collection of a controller $\mathcal{F}_k^c$ and an encoder $\mathcal{F}_k^e$.

E. Problem Statement and M-th Moment Stability

Definition 2.3: (Worst Case Envelope): Let $\mathbf{x}(k)$ be the solution to (8) under a given feedback scheme. Given any realization of the random variables $\mathbf{r}(k)$, $\mathbf{a}(k)$, $G_f(\mathbf{x})(k)$, $\mathbf{z}_a(k)$ and $\mathbf{d}(k)$, the worst case envelope $\bar{\mathbf{x}}(k)$ is the random variable whose realization is defined by

$$\bar{x}(k) = \sup_{x(0)\in\left[-\frac{1}{2},\frac{1}{2}\right]} |x(k)|\,. \tag{13}$$

Consequently, $\bar{\mathbf{x}}(k)$ is the smallest envelope that contains every trajectory generated by an initial condition in the interval $x(0) \in [-(1/2),1/2]$. We adopted the interval $[-(1/2),1/2]$ to make the paper more readable. All results are valid if it is replaced by any other symmetric bounded interval.

Our problem is to determine necessary and sufficient conditions that guarantee the existence of a stabilizing feedback scheme. The results are derived for the following notion of stability.

Definition 2.4: (mth Moment Robust Stability): Let $m > 0$, $\varrho \in \mathbb{N}_+\bigcup\{\infty\}$, $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given. The system (8), under a given feedback scheme, is mth moment (robustly) stable provided that the following holds:

$$\begin{cases} \lim_{k\to\infty}\mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right]=0, & \text{if } \bar{z}_f=\bar{d}=0 \\ \exists b\in\mathbb{R}\in_+ \text{ s.t. } \limsup_{k\to\infty}\mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right]<b, & \text{otherwise} \end{cases} \tag{14}$$

The first limit in (14) is an internal stability condition while the second is an external stability condition. The constant b must be such that $\limsup_{k\to\infty}\mathcal{E}[\bar{\mathbf{x}}(k)^m] < b$ holds for all allowable disturbances $\|\mathbf{d}\|_\infty < d$, as well as every $\mathbf{z}_a$ and $G_f(\mathbf{x})$ satisfying (6), (7).

F. Main Results and Conclusions

The main results of the paper are the sufficiency Theorems 3.2 and 3.4 proved in Section III. The sufficiency conditions are proved constructively by means of the stabilizing feedback scheme of Definition 3.3. The necessary and sufficient conditions can be expressed as inequalities involving $\mathcal{C}$ and $\mathcal{R}$ plus a few auxiliary quantities that depend on the statistical behavior of the plant and the link as well as the descriptions of uncertainty. The intuition behind such auxiliary quantities is given in Section V. In order to preserve stability, the presence of randomness must be offset by an increase of the average transmission rate $\mathcal{C}$. In addition, we find that the higher $\mathcal{C}$ the larger the tolerance to uncertainty in the plant.

III. Sufficiency Conditions for the Robust Stabilization of First-Order Linear Systems

In this section, we derive constructive sufficient conditions for the existence of a stabilizing feedback scheme. We start with the deterministic case in Section III-A, while Section III-B deals with random $\mathbf{r}$ and $\mathbf{a}$. We stress that our proofs hold under the framework of Section II. The following definition introduces the main idea behind the construction of a stabilizing feedback scheme.

Definition 3.1: (Upper-Bound Sequence): Let $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$, $\bar{d} \geq 0$ and $\varrho \in \mathbb{N}_+\bigcup\{\infty\}$ be given. Define the upper-bound sequence as

$$\begin{aligned}\mathbf{v}(k+1) = {} & |\mathbf{a}(k)|\,2^{-\mathbf{r}_e(k)}\mathbf{v}(k) \\ & +\bar{z}_f\max\left\{\mathbf{v}(k-\varrho),\ldots,\mathbf{v}(k)\right\}+\bar{d}\end{aligned} \tag{15}$$

where $v(i)=0$ for $i<0$,$v(0)=1/2$ and $\mathbf{r}_e(k)$ is an effective rate given by

$$\mathbf{r}_e(k) = -\log\left(2^{-\mathbf{r}(k)} + \bar{z}_a\right). \tag{16}$$

Definition 3.2: Following the definition of $\mathbf{r}(k)$ we also define C_e and $\mathbf{r}_e^\delta(k)$ such that:

$$\mathbf{r}_e(k) = C_e - \mathbf{r}_e^\delta(k) \tag{17}$$

where $\mathcal{E}[\mathbf{r}_e^\delta(k)] = 0$.

We adopt $v(0) = 1/2$ to guarantee that $|x(0)| \leq v(0)$. If $x(0) = 0$ then we can select $v(0) = 0$. Notice that the multiplicative uncertainty $\bar{z}_a$ acts by reducing the effective rate $\mathbf{r}_e(k)$. After inspecting (16), we find that $\mathbf{r}_e(k) \leq \min\{\mathbf{r}(k), -\log(\bar{z}_a)\}$. Also, we have that

$$\bar{z}_a = 0 \Longrightarrow \mathbf{r}_e(k) = \mathbf{r}(k),\ \mathbf{r}_e^\delta(k) = \mathbf{r}^\delta(k) \text{ and } C = C_e. \tag{18}$$

Notice that $\mathbf{v}(k)$ can be constructed at the controller and the encoder because both have access to ϱ,$\bar{z}_f$,$\bar{z}_a$, $\bar{d}$ and $(\mathbf{r}(0),\ldots,\mathbf{r}(k-1))$, while $(\mathbf{a}(0),\ldots,\mathbf{a}(k))$ is accessible at the controller (see Section II-D).

Definition 3.3: (Stabilizing Feedback Scheme): The feedback scheme is defined as follows.

- **Encoder**: Measures $x(k)$ and computes $b_i \in \{0,1\}$ such that

$$(b_1,\ldots,b_{\bar{r}}) = arg \max_{\sum_{i=1}^{\bar{r}} b_i \frac{1}{2^i} \leq \left(\frac{\mathbf{x}(k)}{2\mathbf{v}(k)}+\frac{1}{2}\right)} \sum_{i=1}^{\bar{r}} b_i \frac{1}{2^i}, \tag{19}$$

  Place $(b_1,\ldots,b_{\bar{r}})$ for transmission. For any $r(k) \in \{0,\ldots,\bar{r}\}$, the above construction provides the following centroid approximation $\hat{x}(k)$ for $x(k) \in [-v(k), v(k)]$:

$$\hat{x}(k) = 2v(k)\left(\sum_{i=1}^{r(k)} b_i \frac{1}{2^i} + \frac{1}{2^{r(k)+1}} - \frac{1}{2}\right) \tag{20}$$

  which satisfies $|x(k)-\hat{x}(k)| \leq 2^{-r(k)}v(k)$.
- **Controller**: From the $\bar{r}$ bits placed for transmission in the stochastic link, only $\mathbf{r}(k)$ bits go through. Compute $\mathbf{u}(k)$ as

$$\mathbf{u}(k) = -\mathbf{a}(k)\hat{\mathbf{x}}(k), \tag{21}$$

  As expected, the transmission of state information through a finite capacity medium requires quantization. The encoding scheme of Definition 3.3 is not an exception and is structurally identical to the ones used in [2] and [24], where sequences were already used to upper-bound the state of the plant.

The following lemma suggests that, in the construction of stabilizing controllers, we may choose to focus on the dynamics of the sequence $\mathbf{v}(k)$. That simplifies the analysis in the presence of uncertainty because the dynamics of $\mathbf{v}(k)$ is described by a first-order difference equation.

Lemma 3.1: Let $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \leq 0$ be given. If $\mathbf{x}(k)$ is the solution of (8) under the feedback scheme of definition 3.3, then the following holds:

$$\bar{\mathbf{x}}(k) \leq \mathbf{v}(k)$$

for all $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, every choice of $|\mathbf{d}(k)| \leq \bar{d}$, $G_f \in \{G \in \mathbb{G}_\varrho : \|G\|_\infty \leq \bar{z}_f\}$ and $|\mathbf{z}_a(k)| \leq \bar{z}_a$.

Proof: We proceed by induction, assuming that $\bar{x}(i) \leq v(i)$ for $i \in \{0,\ldots,k\}$ and proving that $\bar{x}(k+1) \leq v(k+1)$. From (8), we get

$$\begin{aligned}|x(k+1)| \leq |a(k)| \left| x(k) + \frac{u(k)}{a(k)} \right| + |z_a(k)|\,|a(k)|\,|x(k)| \\ + |G_f(x)(k)| + |d(k)|. \end{aligned} \tag{22}$$

The way the encoder constructs the binary expansion of the state, as well as (21), allow us to conclude that $|x(k) + (u(k)/a(k))| \leq 2^{-r(k)}v(k)$. Now, we recall that $|z_a(k)| \leq \bar{z}_a$, $|G_f(x)(k)| \leq \bar{z}_f \max\{v(k-\varrho),\ldots,v(k)\}$ and that $|d(k)| \leq \bar{d}$, so that (22) implies

$$\begin{aligned}|x(k+1)| \leq |a(k)| \left(2^{-r(k)} + \bar{z}_a\right) v(k) \\ + \bar{z}_f \max\{v(k-\varrho),\ldots,v(k)\} + \bar{d}. \end{aligned} \tag{23}$$

The proof is concluded once we realize that $|x(0)| \leq v(0)$. □

A. The Deterministic Case

We start by deriving a sufficient condition for the existence of a stabilizing feedback scheme in the deterministic case, i.e., $\mathbf{r}(k) = \mathcal{C}$ and $\log(|\mathbf{a}(k)|) = \mathcal{R}$. Subsequently, we move for the stochastic case where we derive a sufficient condition for stabilizability.

Theorem 3.2: (Sufficiency Conditions for Robust Stability): Let $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given and $h(k)$ be defined as

$$h(k) = 2^{k(\mathcal{R}-\mathcal{C}_e)}, \qquad k \geq 0$$

where $\mathcal{C}_e = r_e = -\log(2^{-C} + \bar{z}_a)$.

Consider that $\mathbf{x}(k)$ is the solution of (8) under the feedback scheme of Definition 3.3 as well as the following conditions:

- **C1)** $\mathcal{C}_e > \mathcal{R}$;
- **C2)** $\bar{z}_f \|h\|_1 < 1$.

If **C1)** and **C2)** are satisfied, then the following holds:

$$\bar{x}(k) \leq \|h\|_1 \left(\bar{z}_f \frac{\|h\|_1 \bar{d} + \frac{1}{2}}{1 - \|h\|_1 \bar{z}_f} + \bar{d}\right) + h(k)\frac{1}{2} \tag{24}$$

for all $|\mathbf{d}(k)| \leq \bar{d}$, $G_f \in \{G \in \mathbb{G}\varrho : \|G\|_\infty \leq \bar{z}_f\}$ and $|\mathbf{z}_a(k)| \leq \bar{z}_a$.

Proof: From Definition 3.1, we know that, for arbitrary $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$

$$v(k+1) = 2^{\mathcal{R}-\mathcal{C}_e} v(k) + \bar{z}_f \max\{v(k-\varrho),\ldots,v(k)\} + \bar{d}. \tag{25}$$

Solving the difference equation for $k \geq 1$ gives

$$\begin{aligned} v(k) = 2^{k(\mathcal{R}-\mathcal{C}_e)} v(0) + \sum_{i=0}^{k-1} 2^{(k-i-1)(\mathcal{R}-\mathcal{C}_e)} \\ \times \left(\bar{z}_f \max\{v(i-\varrho),\ldots,v(i)\} + \bar{d}\right) \end{aligned} \tag{26}$$

which, using $\|\prod^k v\|_\infty \stackrel{\text{def}}{=} \max\{v(0),\ldots,v(k)\}$, leads to

$$v(k) \leq \|h\|_1 \left(\bar{z}_f \|\Pi^k v\|_\infty + \bar{d}\right) + 2^{k(\mathcal{R}-\mathcal{C}_e)} v(0). \tag{27}$$

However, we also know that $2^{k(\mathcal{R}-\mathcal{C}_e)} \leq 1$, so that

$$\|\Pi^k v\|_\infty \leq \|h\|_1 \left(\bar{z}_f \|\Pi^k v\|_\infty + \bar{d}\right) + v(0) \tag{28}$$

which implies

$$\|\Pi^k v\|_\infty \le \frac{\|h\|_1 \bar{d} + v(0)}{1 - \|h\|_1 \bar{z}_f}. \quad (29)$$

Direct substitution of (29) in (27) leads to $v(k) \le \|h\|_1(\bar{z}_f((\|h\|_1\bar{d} + v(0))/(1 - \|h\|_1\bar{z}_f)) + \bar{d}) + 2^{k(\mathcal{R}-\mathcal{C}_e)}v(0)$. The proof is complete once we make $v(0) = 1/2$ and use Lemma 3.1 to conclude that $\bar{x}(k) \le v(k)$. □

B. Sufficient Condition for the Stochastic Case

The following Lemma provides a sequence, denoted by $v_m(k)$, which is an upper-bound for the mth moment of $\bar{\mathbf{x}}(k)$. We show that v_m is propagated according to a first-order difference equation that is suitable for the analysis in the presence of uncertainty.

Lemma 3.3: (mth Moment Boundedness): Let $\varrho \in \mathbb{N}_+$, $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$, $\bar{d} \ge 0$ and m be given. Consider the following sequence defined for $k \ge 1$:

$$v_m(k) = h_m(k)v_m(0) + \bar{d}\sum_{i=0}^{k-1} h_m(k-i-1) + (\varrho+1)^{\frac{1}{m}}\bar{z}_f \times \sum_{i=0}^{k-1} h_m(k-i-1)\max\{v_m(i-\varrho),\ldots,v_m(i)\} \quad (30)$$

where $v_m(i) = 0$ for $i < 0$, $v_m(0) = 1/2$, $h_m(k)$ is the impulse response given by:

$$h_m(k) = \left(\mathcal{E}\left[2^{m(\log(|a(k)|) - r_e(k))}\right]\right)^{\frac{k}{m}}, \qquad k \ge 0 \quad (31)$$

and $r_e(k) = -\log(2^{-r(k)} + \bar{z}_a)$. If $\mathbf{x}(k)$ is the solution of (8) under the feedback scheme of Definition 3.3, then the following holds:

$$\mathcal{E}[\bar{\mathbf{x}}(k)^m] \le v_m(k)^m$$

for all $|\mathbf{d}(t)| \le \bar{d}$, $G_f \in \{G_f \in \mathbb{G}_\varrho : \|G_f\|_\infty \le \bar{z}_f\}$ and $|\mathbf{z}_a(k)| \le \bar{z}_a$.

Proof: Since Lemma 3.1 guarantees that $\bar{x}(k+1) \le v(k+1)$, we only need to show that $\mathcal{E}[\mathbf{v}(k+1)^m]^{1/m} \le v_m(k+1)$. Again, we proceed by induction by noticing that $v(0) = v_m(0)$ and by assuming that $\mathcal{E}[\mathbf{v}(i)^m]^{1/m} \le v_m(i)$ for $i \in \{1,\ldots,k\}$. The induction hypothesis is proved once we establish that $\mathcal{E}[\mathbf{v}(k+1)^m]^{1/m} \le v_m(k+1)$. From Definition 3.1, we know that

$$\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} = \mathcal{E}\left[\left(2^{\log(|\mathbf{a}(k)|) - \mathbf{r}_e(k)}\mathbf{v}(k) + \bar{z}_f\max\{\mathbf{v}(k-\varrho),\ldots,\mathbf{v}(k)\} + \bar{d}\right)^m\right]^{\frac{1}{m}} \quad (32)$$

Using Minkowsky's inequality [8] as well as the fact that $\mathbf{v}(i)$ is independent of $\mathbf{a}(j)$ and $\mathbf{r}_e(j)$ for $j \ge i$, we get

$$\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} \le \mathcal{E}\left[2^{m(\log(|\mathbf{a}(k)|) - \mathbf{r}_e(k))}\right]^{\frac{1}{m}}\mathcal{E}[\mathbf{v}(k)^m]^{\frac{1}{m}} + \bar{z}_f\mathcal{E}[\max\{\mathbf{v}(k-\varrho),\ldots,\mathbf{v}(k)\}^m]^{\frac{1}{m}} + \bar{d} \quad (33)$$

which, using the inductive assumption, implies the following inequality:

$$\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} \le \mathcal{E}\left[2^{m(\log(|\mathbf{a}(k)|) - \mathbf{r}_e(k))}\right]^{\frac{1}{m}} v_m(k) + (\varrho+1)^{\frac{1}{m}}\bar{z}_f\max\{v_m(k-\varrho),\ldots,v_m(k)\} + \bar{d} \quad (34)$$

where we used the fact that, for arbitrary random variables $\mathbf{s}_1,\ldots,\mathbf{s}_n$, the following holds:

$$\begin{aligned}&\mathcal{E}[\max\{|\mathbf{s}_1|,\ldots,|\mathbf{s}_n|\}^m] \\ &\quad \le \mathcal{E}\left[\sum_{i=1}^{n}|\mathbf{s}_1|^m\right] \\ &\quad \le n\max\{\mathcal{E}[|\mathbf{s}_1|^m],\ldots,\mathcal{E}[|\mathbf{s}_n|^m]\}. \end{aligned} \quad (35)$$

The proof follows once we notice that the right-hand side of (34) is just $v_m(k+1)$. □

Theorem 3.4: (Sufficient Condition): Let $m,\varrho \in \mathbb{N}_+$, $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \ge 0$ be given along with the quantities that follow:

$$\begin{aligned}\beta(m) &= \frac{1}{m}\log\mathcal{E}\left[2^{m\mathbf{l}_a^\delta(k)}\right] \\ \alpha_e(m) &= \frac{1}{m}\log\mathcal{E}\left[2^{m\mathbf{r}_e^\delta(k)}\right] \\ h_m(k) &= 2^{k(\mathcal{R}+\beta(m)+\alpha_e(m)-\mathcal{C}_e)}, \qquad k \ge 0\end{aligned}$$

where $\mathbf{r}_e^\delta$ and C_e come from (17). Consider that $\mathbf{x}(k)$ is the solution of (8) under the feedback scheme of Definition 3.3 as well as the following conditions:

- **C3)** $C_e > \mathcal{R} + \beta(m) + \alpha_e(m)$;
- **C4)** $(\varrho+1)^{1/m}\bar{z}_f\|h\|_1 < 1$.

If **C3)** and **C4)** are satisfied, then the following holds:

$$\mathcal{E}[\bar{\mathbf{x}}(k)^m]^{\frac{1}{m}} \le h_m(k)\frac{1}{2} + \|h_m\|_1\left((\varrho+1)^{\frac{1}{m}}\bar{z}_f\frac{\|h_m\|_1\bar{d} + \frac{1}{2}}{1 - (\varrho+1)^{\frac{1}{m}}\bar{z}_f\|h_m\|_1} + \bar{d}\right) \quad (36)$$

for all $|\mathbf{d}(k)| \le \bar{d}$, $G_f \in \{G_f \in \mathbb{G}_\varrho : \|G_f\|_\infty \le \bar{z}_f\}$ and $|\mathbf{z}_a(k)| \le \bar{z}_a$.

Proof: Using v_m from Lemma 3.3, we arrive at

$$v_m(k) \le h_m(k)v_m(0) + \|h_m\|_1\left((\varrho+1)^{\frac{1}{m}}\bar{z}_f\|\Pi^k v_m\|_\infty + \bar{d}\right) \quad (37)$$

where we use $\|\prod^k v_m\|_\infty \stackrel{\text{def}}{=} \max\{v_m(0),\ldots,v_m(k)\}$. However, from (37), we conclude that

$$\|\Pi^k v_m\|_\infty \le v_m(0) + \|h_m\|_1\left((\varrho+1)^{\frac{1}{m}}\bar{z}_f\|\Pi^k v_m\|_\infty + \bar{d}\right) \quad (38)$$

or, equivalently

$$\|\Pi^k v_m\|_\infty \le \frac{v_m(0) + \|h_m\|_1\bar{d}}{1 - \|h_m\|_1(\varrho+1)^{\frac{1}{m}}\bar{z}_f}. \quad (39)$$

Substituting (39) in (37), gives

$$v_m(k) \le h_m(k)v_m(0) + \|h_m\|_1\left((\varrho+1)^{\frac{1}{m}}\bar{z}_f\frac{v_m(0) + \|h_m\|_1\bar{d}}{1 - \|h_m\|_1(\varrho+1)^{\frac{1}{m}}\bar{z}_f} + \bar{d}\right). \quad (40)$$

The proof follows from Lemma 3.3 and by noticing that $h_m(k)$ can be rewritten as

$$h_m(k) = \left(\mathcal{E}\left[2^{m(\log(|\mathbf{a}(k)|)-\mathbf{r}_e(k))}\right)^{\frac{k}{m}} = 2^{k(\mathcal{R}+\beta(m)+\alpha_e(m)-\mathcal{C}_e)}. \quad (41)$$

□

C. Extension to Multistate Systems

The multi-state case entails several difficult challenges and the possibility of extending our results has to be studied on a case-by-case basis. The following is a list of problems that have to be faced in such extension.

- If the system is stochastic and multistate, then modal decomposition might not be possible [16].
- Even if the system is time-invariant then one needs to solve an allocation problem. To see that, one has to consider that stabilization must be guaranteed for all modes. An allocation algorithm must regulate the transmission of information about each mode through one link which might also exhibit rate fluctuations.
- The previous questions may get even more complicated in the presence of output feedback.

In [14], we provide examples where the multistate case can be tackled. In particular, we look at uncertain fully observed time-invariant plants. The approach there uses real Jordan forms, in the same manner as [24]. Very particular instances of the stochastic case may also be found there. We stress that the contribution in [14], for the stochastic case, is very modest and leaves open an interesting area for further research. Results for the fully observed Markovian case over finite alphabets, in the presence of a deterministic link, can be found in [16].

IV. Necessary Conditions for the Existence of Stabilizing Feedback Schemes

Consider that $\bar{z}_a = \bar{z}_f = \bar{d} = 0$. We derive necessary conditions for the existence of an internally stabilizing feedback scheme. We emphasize that the proofs in this section use the mth moment stability criteria and that they are valid regardless of the encoding/decoding scheme. They follow from a counting argument[5] which is identical to the one used by [24] and [26]. Necessary conditions for stability were also studied for the Gaussian channel in [27] and for other stochastic channels in [21], [22]. A necessary condition for the almost sure stabilizability in the presence of arbitrary stochastic channels is given by [26]. We include our treatment, because it provides necessary conditions for mth moment stability, which are inequalities involving directly the defined quantities $\alpha(m)$ and $\beta(m)$. Such quantities are an important aid on the derivation of the conclusions presented in Section V.

We derive the necessary condition for the following class of state–space representations:

$$\mathbf{x}(k) = \mathbf{U}(k)\mathbf{x}(k) + B\mathbf{u}(k) \quad (42)$$

[5]We also emphasize that this proof is different from what we had originally. The present argument was suggested by a reviewer of one of our publications

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_b}$, $B \in \mathbb{R}^{n \times n_b}$ and $\mathbf{U}(k)$ is a block upper-triangular matrix of the form

$$\mathbf{U}(k) = \begin{bmatrix} \mathbf{a}(k)\mathbf{Rot}(k) & \cdots & \cdots & \\ 0 & \mathbf{a}(k)\mathbf{Rot}(k) & \ddots & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & \mathbf{a}(k)\mathbf{Rot}(k) \end{bmatrix} \quad (43)$$

and $\mathbf{Rot}$ is a sequence of random rotation matrices satisfying $\det(Rot(k)) = 1$. We also assume that $\mathbf{Rot}$ is independent of $\mathbf{r}$. The motivation for choosing (43) is that $\mathbf{U}(k)$ may be taken as a real Jordan block. The representation (42) encompasses the stochastic first-order case and it is also important in the representation of time-invariant plants. Notice that every dynamic matrix A, of a linear and time-invariant system, is similar to a real Jordan form, i.e., a block diagonal matrix where each block is a real Jordan block [9].

In this section, we also adopt the following mth order generalization of $\bar{\mathbf{x}}(k)$:

$$\bar{x}(k) \stackrel{\text{def}}{=} \sup_{x(0)\in\left[-\frac{1}{2},\frac{1}{2}\right]^n} \max_{i\in\{1,\ldots,n\}} |[x(k)]_i|. \quad (44)$$

Theorem 4.1: Let $\mathbf{x}(k)$ be the solution of the state–space equation (42) along with $\alpha(z)$ and $\beta(z)$ given by

$$\alpha(z) = \frac{1}{z}\log\left(\mathcal{E}\left[2^{z\mathbf{r}^{\delta}(k)}\right]\right), \qquad z > 0 \quad (45)$$

$$\beta(z) = \frac{1}{z}\log\left(\mathcal{E}\left[2^{z\mathbf{l}_a^{\delta}(k)}\right]\right), \qquad z > 0. \quad (46)$$

If the feedback system is mth moment stable, i.e., $\lim_{k\to\infty}\mathcal{E}[\bar{\mathbf{x}}(k)^m] = 0$ then the following inequality is satisfied:

$$C - \alpha\left(\frac{m}{n}\right) > n\beta(m) + nR. \quad (47)$$

If we just require $\sup_k \mathcal{E}[\bar{x}(k)^m] < \infty$, then the following must hold:

$$C - \alpha\left(\frac{m}{n}\right) \geq n\beta(m) + nR. \quad (48)$$

Proof: (See [26] for more details on similar counting arguments)

Consider a specific realization of $\mathbf{Rot}$, $\mathbf{r}$ and $\mathbf{a}$ along with the following sets:

$$\bar{\Omega}_k \stackrel{\text{def}}{=} \left\{\Pi_{i=0}^{k-1}U(i)x(0) : x(0) \in \left[-\frac{1}{2},\frac{1}{2}\right]^n\right\} \quad (49)$$

$$\Omega_k\left(\{u(i)\}_{i=0}^{k-1}\right) \stackrel{\text{def}}{=} \left\{x(k) : \{u(i)\}_{i=0}^{k-1} = \mathcal{F}(x(0),k), x(0) \in \left[-\frac{1}{2},\frac{1}{2}\right]^n\right\} \quad (50)$$

where $u(k)$ is obtained through a fixed feedback law $\mathcal{F}(x(0),k)$. Since $x(k)$ is given by (42) and $\{u(i)\}_{i=0}^{k-1}$ can take, at most, $2^{\sum_{i=0}^{k-1} r(i)}$ values, we find that

$$\frac{Vol(\bar{\Omega}_k)}{\max_{\{u(i)\}_{i=0}^{k-1}} Vol\left(\Omega_k\left(\{u(i)\}_{i=0}^{k-1}\right)\right)} \leq 2^{\sum_{i=0}^{k-1} r(i)}. \quad (51)$$

Consequently, using (51) we infer that

$$2^{\sum_{i=0}^{k-1}\log|det(U(i))|}2^{-\sum_{i=0}^{k-1}r(i)} \leq 2^n \bar{x}(k)^n \tag{52}$$

where we used $Vol(\bar{\Omega}_k) = 2^{\sum_{i=0}^{k-1}\log|det(U(i))|}$ and $(2\bar{x}(k))^n \geq Vol(\Omega_k(\{u(i)\}_{i=0}^{k=1}))$. By taking expectations, the mth moment stability assumption leads to

$$\lim_{k\to\infty}\left(\mathcal{E}\left[2^{\frac{m}{n}\log|det(\mathbf{U}(k))|}\right]\mathcal{E}\left[2^{-\frac{m}{n}\mathbf{r}(k)}\right]\right)^k = 0$$

which concludes the proof. The proof for the case where only $\sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$ is required also follows directly from the previous analysis. □

Corollary 4.2: Let $\mathbf{x}(k)$ be the solution of the following linear and time-invariant system:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k). \tag{53}$$

If the feedback system is mth moment stable, i.e., $\lim_{k\to\infty}\mathcal{E}[\bar{\mathbf{x}}(k)^m] = 0$ then the following inequality is satisfied:

$$C - \alpha\left(\frac{m}{n_{unstable}}\right) > \sum_{i=1}^{n}\max\{\log|\lambda_i(A)|, 0\} \tag{54}$$

where $n_{unstable}$ is the number of unstable eigenvalues of A. If we just require $\sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$ then the following must hold:

$$C - \alpha\left(\frac{m}{n_{unstable}}\right) \geq \sum_{i=1}^{n}\max\{\log|\lambda_i(A)|, 0\}. \tag{55}$$

Proof: By writing A in its real Jordan form, the proof is a direct adaptation of the proof of Theorem 4.1. □

V. Properties of the Measures $\alpha(m)$ and $\beta(m)$

Consider that $\mathbf{a}$ and $\mathbf{r}$ are stochastic processes, that there is no uncertainty in the plant and no external disturbances, i.e., $\bar{z}_f = \bar{z}_a = \bar{d} = 0$. In such situation, (8) can be written as

$$\mathbf{x}(k+1) = \mathbf{a}(k)\mathbf{x}(k) + \mathbf{u}(k). \tag{56}$$

For a given m, the stability condition of Definition 2.4 becomes

$$\lim_{k\to\infty}\mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right] = 0. \tag{57}$$

If $\mathbf{v}$ is a real random variable then Jensen's inequality [5] implies:

$$\mathcal{E}[2^{\mathbf{v}}] \geq 2^{\mathcal{E}[\mathbf{v}]}$$

where equality is attained if and only if $\mathbf{v}$ is a deterministic constant. As such, $\log(\mathcal{E}[2^{\mathbf{v}}]2^{-\mathcal{E}[\mathbf{v}]}) \geq 0$ can be used as a measure of "randomness" which can be taken as an alternative to variance. Notice that such quantity may be more informative than variance because it depends on higher moments of $\mathbf{v}$. We use this concept to interpret our results and express our conditions in a way that is amenable to a direct comparison with other publications. Along these lines, the following are randomness measures for $\log(|\mathbf{a}(k)|)$ and $\mathbf{r}(k)$:

$$\beta(m) = \frac{1}{m}\log\left(\mathcal{E}\left[2^{m\mathbf{l}_a^\delta(k)}\right]\right) \tag{58}$$

$$\alpha(m) = \frac{1}{m}\log\left(\mathcal{E}\left[2^{m\mathbf{r}^\delta(k)}\right]\right) \tag{59}$$

where $\mathbf{l}_a^\delta(k)$ and $\mathbf{r}^\delta(k)$ are given by (repeated here for convenience)

$$\log(|\mathbf{a}(k)|) = \mathcal{E}\left[\log(|\mathbf{a}(k)|)\right] + \mathbf{l}_a^\delta(k) = R + \mathbf{l}_a^\delta(k) \tag{60}$$

$$\mathbf{r}(k) = \mathcal{E}\left[\mathbf{r}(k)\right] - \mathbf{r}^\delta(k) = C - \mathbf{r}^\delta(k). \tag{61}$$

The following equivalence is a direct consequence of the necessary and sufficient conditions proved in Theorems 4.1 and 3.4:

$$\left(\text{Exists feedback s.t. } \lim_{k\to\infty}\mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right] = 0\right) \Longleftrightarrow \mathcal{C} > \mathcal{R} + \alpha(m) + \beta(m). \tag{62}$$

After examining (62), we infer that $\alpha(m)$ and $\beta(m)$ encompass the influence of m on the stability condition, while C and R are independent of m. Condition (62) suggests that $\alpha(m)$ is the right intuitive measure of quality, of a direct link, for the framework considered in this paper. The following are properties of $\alpha(m)$ and $\beta(m)$.

- Note that Jensen's inequality implies that $\alpha(m) \geq 0$ and $\beta(m) \geq 0$, where equality is achieved only if the corresponding random variable is deterministic. Accordingly, (62) shows that randomness in $\mathbf{r}(k)$ implies that $\mathcal{C} > R + \alpha(m)$ is necessary for stabilization. The fact that randomness in the link creates the need for capacity larger than R, was already established, but quantified differently, in [21]. In addition, we find that randomness in the system adds yet another factor $\beta(m)$.
- By means of a Taylor expansion and taking limits, we get

$$\lim_{m\searrow 0}\alpha(m) = \lim_{m\searrow 0}\beta(m) = 0. \tag{63}$$

  Under the above limit, the necessary and sufficient condition (62) becomes $C > R$. That is the weakest condition of stability and coincides with the one derived by [24] for almost sure stability. By means of (62) and (63), we can also conclude that if $C > R$, i.e. the feedback scheme is almost surely stabilizable [24], then it is mth moment stabilizable for some $m > 0$.
- The opposite limiting case gives

$$\lim_{m\to\infty}\alpha(m) = C - r_{\min} \tag{64}$$

$$\lim_{m\to\infty}\beta(m) = \log(a^{sup}) - R \tag{65}$$

  where

$$r_{\min} = \min\{r \in \{0,\ldots,\bar{r}\} : \mathcal{P}(\mathbf{r}(k) = r) \neq 0\}$$
$$a^{sup} = \sup\{\tilde{a} : \mathcal{P}(|\mathbf{a}(k)| \geq \tilde{a}) \neq 0\}.$$

- $\alpha(m)$ and $\beta(m)$ are nondecreasing functions of m.

From the previous properties of $\alpha(m)$ and $\beta(m)$, we find that the following hold true.

- A feedback scheme is stabilizable for all moments, i.e., $\forall m,\ \sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$ if and only if $r_{\min} > \log(a^{sup})$.
- If $r_{\min} = 0$ then there exists m_0 such that $\forall m \geq m_0$, $\sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$. This is the case of the erasure channel suggested by [21]. (see Example 5.2).
- Similarly, if $\log(a^{sup}) = \infty$ then there exists m_0 such that $\forall m \geq m_0$,$\sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] = \infty$ (see Example 5.1). Notice that a^{sup} can be larger than one and still $\mathcal{E}[\mathbf{a}(k)^m] < 1$ for some m. Even more, in Example 5.1, we have $a^{sup} = \infty$ and $\mathcal{E}[|\mathbf{a}(k)|^m] < \infty$ for all m.

Example 5.1: Consider that $|\mathbf{a}|$ is log-normally distributed, i.e., $\log|\mathbf{a}(k)|$ is normally distributed. An example where $\mathbf{a}(k)$ is log-normally distributed is given by [3]. If $Var(|\mathbf{a}(k)|)$ is the variance of $|\mathbf{a}(k)|$, then $\beta(m)$ is given by

$$\beta(m) = \frac{m}{2}\log\left(1 + \frac{Var\left(|\mathbf{a}(k)|\right)}{\left(\mathcal{E}\left[|\mathbf{a}(k)|\right]\right)^2}\right) \tag{66}$$

where the expression is obtained by direct integration. Note that $\beta(m)$ grows linearly with m. It illustrates a situation where, given $Var(|\mathbf{a}(k)|) > 0, \mathcal{C}$ and $\alpha(m)$, there always exist m large enough such that the necessary and sufficient condition $\mathcal{C} > \mathcal{R} + \beta(m) + \alpha(m)$ is violated.

The above analysis stresses the fact that feedback, using a direct link, acts by increasing $m^{\max}$ for which $\forall m \leq m^{\max}$, $\sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$ is satisfied. In some cases one may get $m^{\max} = \infty$.

1) The Exponential Statistic: Directly from (58) and (59), we derive the equivalence

$$\mathcal{C} > \mathcal{R} + \alpha(m) + \beta(m) \Longleftrightarrow \mathcal{E}\left[|\mathbf{a}(k)|^m\right]\mathcal{E}\left[2^{-m\mathbf{r}(k)}\right] < 1. \tag{67}$$

The equivalences expressed in (62) and (67) show that all the information we need to know about the link is $\alpha(m)$ and $\mathcal{C}$ or, equivalently, $\mathcal{E}[2^{-m\mathbf{r}(k)}]$.

Example 5.2: (From [21]) The binary erasure channel is a particular case of the class of direct links considered. It can be described by taking $r(k) = 1$ with probability $1 - p_e$ and $r(k) = 0$ with probability of erasure p_e. In that case, $\mathcal{E}[2^{-m\mathbf{r}(k)}] = 2^{-m}(1-p_e)+p_e$. After working through the formulas, one may use (67) and (62) to get the same result as in [21]. In particular, the necessary and sufficient condition for the existence of a stabilizing feedback, for the time-invariant system with $\mathbf{a}(k) = a$, is given by

$$0 \leq p_e < 1 - \frac{|a|^m - 1}{|a|^m(1 - 2^{-m})}$$

A. Determining the Decay of the Probability Distribution Function of $\bar{\mathbf{x}}$

In this section, we explore (62) as a way to infer the decay of the probability distribution of $\bar{\mathbf{x}}(k)$. From Markov's inequality [1, p. 80], we have that

$$\forall m > 0,\ \forall k \geq 0,\ \mathcal{P}\left(\bar{\mathbf{x}}(k) > \vartheta\right) \leq \vartheta^{-m}\mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right]. \tag{68}$$

On the other hand, for any given m, if $\bar{\mathbf{x}}(k)$ has a probability density function then

$$\exists\varepsilon,\ \delta > 1,\ \forall k \geq 0,\ \forall\vartheta > 0,\ \mathcal{P}\left(\bar{\mathbf{x}}(k) > \vartheta\right) \leq \varepsilon\vartheta^{-(m+\delta)} \Longrightarrow \limsup_{k\to\infty}\mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right] < \infty. \tag{69}$$

As such, we infer that (62) and (68), (69) lead to

$$\mathcal{C} > \mathcal{R} + \alpha(m) + \beta(m) \Longrightarrow \exists\varepsilon > 0,\ \forall k \geq 0,\ \forall\vartheta > 0\ \ \mathcal{P}\left(\bar{\mathbf{x}}(k) > \vartheta\right) \leq \varepsilon\vartheta^{-m} \tag{70}$$

$$\mathcal{C} < \mathcal{R} + \alpha(m) + \beta(m) \Longrightarrow \forall\varepsilon > 0,\ \forall\delta > 1,\ \exists k \geq 0\ \ \exists\vartheta > 0,\ \mathcal{P}\left(\bar{\mathbf{x}}(k) > \vartheta\right) > \varepsilon\vartheta^{-(m+\delta)}. \tag{71}$$

B. Uncertainty Interpretation of the Statistical Description of the Direct Link

We suggest that $\alpha(m)$ can be viewed not only as a measure of the quality of the link, in the sense of how $\mathbf{r}(k)$ is expected to fluctuate over time, but it can also be modified to encapsulate a description of uncertainty. To be more precise, consider that Δ_l is an uncertainty set of direct links and that the "nominal" link has a deterministic data-rate $\mathbf{r}^o(k) = \mathcal{C}$. The elements of Δ_l are the following probability mass functions:

$$\Delta_l \subset \left\{p_l \in [0,1]^{\bar{r}+1} : \sum_{i=0}^{\bar{r}}[p_l]_{i+1} = 1, \sum_{i=0}^{\bar{r}} i \times [p_l]_{i+1} = \mathcal{C}\right\}$$

where $p_l \in \Delta_l$ represents a direct link by specifying its statistics, i.e., $\mathcal{P}(r(k) = i) = [p_l]_{i+1}$. The following is a measure of uncertainty in the link:

$$\bar{\alpha}(m) = \sup_{p_l\in\Delta_l}\frac{1}{m}\log\left(\mathcal{E}\left[2^{m\mathbf{r}^\delta(k)}\right]\right). \tag{72}$$

In this situation, (62) implies that the following is a necessary and sufficient condition for the existence of a feedback scheme that is stabilizing for all direct links in the uncertainty set Δ_l

$$\mathcal{C} - \mathcal{R} - \beta(m) > \bar{\alpha}(m).$$

C. Issues on the Stabilization of Linearizable Nonlinear Systems

In this section, we explain why a minimum rate must be guaranteed at all times in order to achieve stabilization in the sense of Lyapunov.[6] Consider that the following is a statespace representation which corresponds to the linearization of a nonlinear system around an equilibrium point:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \quad \mathbf{y}(k) = C\mathbf{x}(k). \tag{73}$$

Consequently, stability in the sense of Lyapunov implies that

$$\sup_k \sup_{x(0)\in\left[-\frac{1}{2},\frac{1}{2}\right]^n} \|\boldsymbol{x}(k)\|_\infty < \infty \tag{74}$$

where $\|x(k)\|_\infty = \max_i |[x(k)]_i|$ and $[x(k)]_i$ are the components of $x(k)$. However, (74) implies that $\mathbf{x}(k)$ is stable for all moments, so Corollary 4.2 leads to:

$$\forall m \geq 0,\ C - \alpha\left(\frac{m}{n_{\text{unstable}}}\right) \geq \sum_i \max\left\{\log|\lambda_i(A)|, 0\right\} \tag{75}$$

[6] Also denoted as $\epsilon - \delta$ stability

which, by means of (64), also implies that $r_{\min} \geq \sum_i \max\{\log|\lambda_i(A)|, 0\}$. As a consequence, local stabilization imposes a minimum rate which has to be guaranteed at all times. The classical packet-erasure channel is characterized by $r_{\min} = 0$ and, as such, it cannot be used for stabilization in the sense of Lyapunov. The fact that the classical erasure channel cannot be used to achieve stability in the sense of Lyapunov could already be inferred from [22].

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Nuno C. Martins (M'05) received the Licenciatura and M.S. degrees in electrical engineering from Instituto Superior Tecnico, Lisbon, Portugal, in 1994 and 1997, respectively, and the Ph.D. degree in electrical engineering and computer science from Massachusetts Institute of Technology (MIT), Cambridge, in 2004.

Currently, he is an Assistant Professor at the Department of Electrical and Computer Engineering, University of Maryland, College Park. He is also affiliated with the Institute for Systems Research. His research interests include fundamental limits of feedback and the fusion between control theory and information theory.

Munther A. Dahleh (S'84–M'97–SM'97–F'01) received the B.S. degree from Texas A&M University, College Station, and the Ph.D. degree from Rice University, Houston, TX, both in electrical engineering, in 1983 and 1987, respectively.

Since 1987, he has been with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, where he is now a Full Professor.

Dr. Dahleh was the recipient of the Ralph Budd Award for the best thesis at Rice University in 1987, the George Axelby Outstanding Paper Award, a National Science Foundation Presidential Young Investigator Award in 1991, the Finmeccanica Career Development Chair in 1992, the Donald P. Eckman Award from the American Control Council in 1993, and the Graduate Students Council Teaching Award in 1995.

Nicola Elia (M'02) received the Laurea degree in electrical engineering from Politecnico of Turin, Turin, Italy, in 1987, and the Ph.D. degree in electrical engineering and computer science from the Massachusetts Institute of Technology (MIT), Cambridge, in 1996.

He was with the Fiat Research Center, Turin, Italy, from 1987 to 1990. He was a Postdoctoral Associate at the Laboratory for Information and Decision Systems at MIT from 1996 to 1999. Currently, he is an Assistant Professor at the Department of Electrical and Computer Engineering, Iowa State University, Ames. His research interests include computational methods for controller design, complex organizations of systems, hybrid systems, and control with communication constraints.

Dr. Elia received the National Science Foundation CAREER Award in 2001.